NAG Fortran Library Routine Document

S21CBF

Note: before using this routine, please read the Users' Note for your implementation to check the interpretation of ***bold italicised*** terms and other implementation-dependent details.

1 Purpose

S21CBF evaluates the Jacobian elliptic functions sn$z$, cn$z$ and dn$z$ for a complex argument z.

2 Specification

```
SUBROUTINE S21CBF(Z, AK2, SN, CN, DN, IFAIL)
INTEGER          IFAIL
real             AK2
complex          Z, SN, CN, DN
```

3 Description

This routine evaluates the Jacobian elliptic functions $\mathrm{sn}(z|k)$, $\mathrm{cn}(z|k)$ and $\mathrm{dn}(z|k)$ given by

$$\begin{aligned} \mathrm{sn}(z|k) &= \sin\phi \\ \mathrm{cn}(z|k) &= \cos\phi \\ \mathrm{dn}(z|k) &= \sqrt{1-k^2\sin^2\phi}, \end{aligned}$$

where z is a complex argument, k is a real parameter (the *modulus*) with $k^2 \leq 1$ and ϕ (the *amplitude* of z) is defined by the integral

$$z = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

The above definitions can be extended for values of $k^2 > 1$ (see Salzer (1962)) by means of the formulae

$$\begin{aligned} \mathrm{sn}(z|k) &= k_1\mathrm{sn}(kz|k_1) \\ \mathrm{cn}(z|k) &= \mathrm{dn}(kz|k_1) \\ \mathrm{dn}(z|k) &= \mathrm{cn}(kz|k_1), \end{aligned}$$

where $k_1 = 1/k$.

Special values include

$$\begin{aligned} \mathrm{sn}(z|0) &= \sin z \\ \mathrm{cn}(z|0) &= \cos z \\ \mathrm{dn}(z|0) &= 1 \\ \mathrm{sn}(z|1) &= \tanh z \\ \mathrm{cn}(z|1) &= \mathrm{sech}\, z \\ \mathrm{dn}(z|1) &= \mathrm{sech}\, z. \end{aligned}$$

These functions are often simply written as sn$z$, cn$z$ and dn$z$, thereby avoiding explicit reference to the parameter k. They can also be expressed in terms of Jacobian theta functions (see S21CCF).

Another nine elliptic functions may be computed via the formulae

$$\begin{aligned}
\mathrm{cd}z &= \mathrm{cn}z/\mathrm{dn}z \\
\mathrm{sd}z &= \mathrm{sn}z/\mathrm{dn}z \\
\mathrm{nd}z &= 1/\mathrm{dn}z \\
\mathrm{dc}z &= \mathrm{dn}z/\mathrm{cn}z \\
\mathrm{nc}z &= 1/\mathrm{cn}z \\
\mathrm{sc}z &= \mathrm{sn}z/\mathrm{cn}z \\
\mathrm{ns}z &= 1/\mathrm{sn}z \\
\mathrm{ds}z &= \mathrm{dn}z/\mathrm{sn}z \\
\mathrm{cs}z &= \mathrm{cn}z/\mathrm{sn}z
\end{aligned}$$

(see Abramowitz and Stegun (1972)).

The values of sn$z$, cn$z$ and dn$z$ are obtained by calls to S21CAF. Further details can be found in Section 8.

4 References

Abramowitz M and Stegun I A (1972) *Handbook of Mathematical Functions* (3rd Edition) Dover Publications

Salzer H E (1962) Quick calculation of Jacobian elliptic functions *Comm. ACM* **5** 399

5 Parameters

1: Z – ***complex*** *Input*

On entry: the argument z of the functions.

Constraints:

ABS(Re(Z)) $\leq \sqrt{\lambda}$,
ABS(Im(Z)) $\leq \sqrt{\lambda}$, where $\lambda = 1/\mathrm{X02AMF}$.

2: AK2 – ***real*** *Input*

On entry: the value of k^2.

Constraint: $0.0 \leq \mathrm{AK2} \leq 1.0$.

3: SN – ***complex*** *Output*
4: CN – ***complex*** *Output*
5: DN – ***complex*** *Output*

On exit: the values of the functions sn$z$, cn$z$ and dn$z$, respectively.

6: IFAIL – INTEGER *Input/Output*

On entry: IFAIL must be set to 0, -1 or 1. Users who are unfamiliar with this parameter should refer to Chapter P01 for details.

On exit: IFAIL $= 0$ unless the routine detects an error (see Section 6).

For environments where it might be inappropriate to halt program execution when an error is detected, the value -1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, for users not familiar with this parameter the recommended value is 0. **When the value -1 or 1 is used it is essential to test the value of IFAIL on exit.**

6 Error Indicators and Warnings

If on entry IFAIL $= 0$ or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

IFAIL $= 1$

On entry, AK2 < 0.0,
or AK2 > 1.0,
or ABS(Re(Z)) $> \sqrt{\lambda}$,
or ABS(Im(Z)) $> \sqrt{\lambda}$, where $\lambda = 1/\mathrm{X02AMF}$.

7 Accuracy

In principle the routine is capable of achieving full relative precision in the computed values. However, the accuracy obtainable in practice depends on the accuracy of the Fortran intrinsic functions for elementary functions such as SIN and COS.

8 Further Comments

The values of sn$z$, cn$z$ and dn$z$ are computed via the formulae

$$\begin{aligned}
\mathrm{sn}z &= \frac{\mathrm{sn}(u,k)\mathrm{dn}(v,k')}{1-\mathrm{dn}^2(u,k)\mathrm{sn}^2(v,k')} + i\frac{\mathrm{cn}(u,k)\mathrm{dn}(u,k)\mathrm{sn}(v,k')\mathrm{cn}(v,k')}{1-\mathrm{dn}^2(u,k)\mathrm{sn}^2(v,k')}\\
\mathrm{cn}z &= \frac{\mathrm{cn}(u,k)\mathrm{cn}(v,k')}{1-\mathrm{dn}^2(u,k)\mathrm{sn}^2(v,k')} - i\frac{\mathrm{sn}(u,k)\mathrm{dn}(u,k)\mathrm{sn}(v,k')\mathrm{dn}(v,k')}{1-\mathrm{dn}^2(u,k)\mathrm{sn}^2(v,k')}\\
\mathrm{dn}z &= \frac{\mathrm{dn}(u,k)\mathrm{cn}(v,k')\mathrm{dn}(v,k')}{1-\mathrm{dn}^2(u,k)\mathrm{sn}^2(v,k')} - i\frac{k^2\mathrm{sn}(u,k)\mathrm{cn}(u,k)\mathrm{sn}(v,k')}{1-\mathrm{dn}^2(u,k)\mathrm{sn}^2(v,k')},
\end{aligned}$$

where $z = u + iv$ and $k' = \sqrt{1-k^2}$ (the *complementary modulus*).

9 Example

The example program evaluates sn$z$, cn$z$ and dn$z$ at $z = -2.0 + 3.0i$ when $k = 0.5$, and prints the results.

9.1 Program Text

Note: the listing of the example program presented below uses ***bold italicised*** terms to denote precision-dependent details. Please read the Users' Note for your implementation to check the interpretation of these terms. As explained in the Essential Introduction to this manual, the results produced may not be identical for all implementations.

```
*     S21CBF Example Program Text.
*     Mark 20 Release. NAG Copyright 2001.
*     .. Parameters ..
      INTEGER          NIN, NOUT
      PARAMETER        (NIN=5,NOUT=6)
*     .. Local Scalars ..
      complex          CN, DN, SN, Z
      real             AK2
      INTEGER          IFAIL
*     .. External Subroutines ..
      EXTERNAL         S21CBF
*     .. Executable Statements ..
      WRITE (NOUT,*) 'S21CBF Example Program Results'
*     Skip heading in data file
      READ (NIN,*)
   20 WRITE (NOUT,*)
      READ (NIN,*,END=40) Z, AK2
```

```
      IFAIL = 0
*
      CALL S21CBF(Z,AK2,SN,CN,DN,IFAIL)
*
      WRITE (NOUT,*) '          Z                    AK2',
     +  '                IFAIL'
      WRITE (NOUT,99999) Z, AK2, IFAIL
      WRITE (NOUT,*)
      WRITE (NOUT,*) '         SN                     CN',
     +  '                   DN'
      WRITE (NOUT,99998) SN, CN, DN
      GO TO 20
   40 STOP
*
99999 FORMAT (1X,'(',F8.4,',',F8.4,')',5X,F10.2,7X,I16)
99998 FORMAT (3(' (',F8.4,',',F8.4,')',3X))
      END
```

9.2 Program Data

```
S21CBF Example Program Data
 (-2.0, 3.0)  0.25 : Values of Z and AK2
```

9.3 Program Results

```
 S21CBF Example Program Results

          Z                    AK2                 IFAIL
 ( -2.0000,  3.0000)          0.25                     0

         SN                     CN                      DN
 ( -1.5865,  0.2456)    (  0.3125,  1.2468)    ( -0.6395, -0.1523)
```