NAG Fortran Library Routine Document

D03NEF

Note: before using this routine, please read the Users' Note for your implementation to check the interpretation of ***bold italicised*** terms and other implementation-dependent details.

1 Purpose

D03NEF computes average values of a continuous function of time over the remaining life of an option. It is used together with D03NDF to value options with time-dependent parameters.

2 Specification

```
SUBROUTINE D03NEF(T0, TMAT, NTD, TD, PHID, PHIAV, WORK, LWORK, IFAIL)
INTEGER           NTD, LWORK, IFAIL
real              T0, TMAT, TD(NTD), PHID(NTD), PHIAV(3), WORK(LWORK)
```

3 Description

D03NEF computes the quantities

$$\phi(t_0), \qquad \hat{\phi} = \tfrac{1}{T-t_0}\int_{t_0}^{T} \phi(\zeta)\, d\zeta, \qquad \bar{\phi} = \left(\tfrac{1}{T-t_0}\int_{t_0}^{T} \phi^2(\zeta)\, d\zeta\right)^{1/2}$$

from a given set of values PHID of a continuous time-dependent function $\phi(t)$ at a set of discrete points TD, where t_0 is the current time and T is the maturity time. Thus $\hat{\phi}$ and $\bar{\phi}$ are first and second order averages of ϕ over the remaining life of an option.

The routine may be used in conjunction with D03NDF in order to value an option in the case where the risk-free interest rate r, the continuous dividend q, or the stock volatility σ is time-dependent and is described by values at a set of discrete times (see Section 8.2). This is illustrated in Section 9.

4 References

None.

5 Parameters

1: T0 – ***real*** *Input*

On entry: the current time t_0.

Constraint: $\mathrm{TD}(1) \leq \mathrm{T0} \leq \mathrm{TD}(\mathrm{NTD})$.

2: TMAT – ***real*** *Input*

On entry: the maturity time T.

Constraint: $\mathrm{TD}(1) \leq \mathrm{TMAT} \leq \mathrm{TD}(\mathrm{NTD})$.

3: NTD – INTEGER *Input*

On entry: the number of discrete times at which ϕ is given.

Constraint: $\mathrm{NTD} \geq 2$.

4: TD(NTD) – ***real*** array *Input*

On entry: the discrete times at which ϕ is specified.

Constraint: $\mathrm{TD}(1) < \mathrm{TD}(2) < \cdots < \mathrm{TD}(\mathrm{NTD})$.

5: PHID(NTD) – ***real*** array *Input*

On entry: PHID(i) must contain the value of ϕ at time TD(i), for $i = 1, 2, \ldots,$ NTD.

6: PHIAV(3) – ***real*** array *Output*

On exit: PHIAV(1) contains the value of ϕ interpolated to t_0, PHIAV(2) contains the first-order average $\hat{\phi}$ and PHIAV(3) contains the second-order average $\bar{\phi}$, where:

$$\hat{\phi} = \frac{1}{T-t_0}\int_{t_0}^{T} \phi(\zeta)\, d\zeta, \qquad \bar{\phi} = \left(\frac{1}{T-t_0}\int_{t_0}^{T} \phi^2(\zeta)\, d\zeta\right)^{1/2}.$$

7: WORK(LWORK) – ***real*** array *Workspace*

8: LWORK – INTEGER *Input*

On entry: the dimension of the array WORK as declared in the (sub)program from which D03NEF is called.

Constraint: LWORK $\geq 9 \times$ NTD $+ 24$.

9: IFAIL – INTEGER *Input/Output*

On entry: IFAIL must be set to 0, -1 or 1. Users who are unfamiliar with this parameter should refer to Chapter P01 for details.

On exit: IFAIL $= 0$ unless the routine detects an error (see Section 6).

For environments where it might be inappropriate to halt program execution when an error is detected, the value -1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, for users not familiar with this parameter the recommended value is 0. **When the value -1 or 1 is used it is essential to test the value of IFAIL on exit.**

6 Error Indicators and Warnings

If on entry IFAIL $= 0$ or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

IFAIL $= 1$

On entry, T0 lies outside the range [TD(1), TD(NTD)],
or TMAT lies outside the range [TD(1), TD(NTD)],
or NTD < 2,
or TD badly ordered,
or LWORK $< 9 \times$ NTD $+ 24$.

IFAIL $= 2$

Unexpected failure in internal call to E01BAF or E02BBF.

7 Accuracy

If $\phi \in C^4[t_0, T]$ then the error in the approximation of $\phi(t_0)$ and $\hat{\phi}$ is $O(H^4)$, where $H = \max_i(\mathrm{T}(i+1) - \mathrm{T}(i))$, for $i = 1, 2, \ldots,$ NTD $- 1$. The approximation is exact for polynomials of degree up to 3.

The third quantity $\bar{\phi}$ is $O(H^2)$, and exact for linear functions.

8 Further Comments

8.1 Timing

The time taken by the routine is proportional to NTD.

8.2 Use with D03NDF

Suppose you wish to evaluate the analytic solution of the Black-Scholes equation in the case when the risk-free interest rate r is a known function of time, and is represented as a set of values at discrete times. A call to D03NEF providing these values in PHID produces an output array PHIAV suitable for use as the argument R in a subsequent call to D03NDF.

Time-dependent values of the continuous dividend Q and the volatility σ may be handled in the same way.

8.3 Algorithmic Details

The NTD data points are fitted with a cubic B-spline using the routine E01BAF. Evaluation is then performed using E02BBF, and the definite integrals are computed using direct integration of the cubic splines in each interval. The special case of $T = t_o$ is handled by interpolating ϕ at that point.

9 Example

This example demonstrates the use of the routine in conjunction with D03NDF to solve the Black-Scholes equation for valuation of a 5-month American call option on a non-dividend-paying stock with an exercise price of \$50. The risk-free interest rate varies linearly with time and the stock volatility has a quadratic variation. Since these functions are integrated exactly by D03NEF the solution of the Black-Scholes equation by D03NDF is also exact.

The option is valued at a range of times and stock prices.

9.1 Program Text

Note: the listing of the example program presented below uses ***bold italicised*** terms to denote precision-dependent details. Please read the Users' Note for your implementation to check the interpretation of these terms. As explained in the Essential Introduction to this manual, the results produced may not be identical for all implementations.

```
*     D03NEF Example Program Text
*     Mark 20 Release. NAG Copyright 2001.
*     .. Parameters ..
      INTEGER          NIN, NOUT
      PARAMETER        (NIN=5,NOUT=6)
      INTEGER          NSMAX, NTMAX, NTDMAX, LWMAX
      PARAMETER        (NSMAX=100,NTMAX=100,NTDMAX=10,LWMAX=9*NTDMAX+24)
*     .. Local Scalars ..
      real             DS, DT, TMAT, X
      INTEGER          I, IFAIL, IGREEK, J, KOPT, LWORK, NS, NT, NTD
*     .. Local Arrays ..
      real             DELTA(NSMAX,NTMAX), F(NSMAX,NTMAX),
     +                 GAMMA(NSMAX,NTMAX), LAMBDA(NSMAX,NTMAX), Q(3),
     +                 RA(3), RD(NTDMAX), RHO(NSMAX,NTMAX), S(NSMAX),
     +                 SIGA(3), SIGD(NTDMAX), T(NTMAX), TD(NTDMAX),
     +                 THETA(NSMAX,NTMAX), WORK(LWMAX)
      LOGICAL          GPRNT(5), TDPAR(3)
      CHARACTER*6      GNAME(5)
*     .. External Subroutines ..
      EXTERNAL         D03NDF, D03NEF
*     .. Intrinsic Functions ..
      INTRINSIC        real
*     .. Data statements ..
      DATA             GNAME/'Theta ', 'Delta ', 'Gamma ', 'Lambda',
     +                 'Rho   '/
      DATA             GPRNT/5*.TRUE./
*     .. Executable Statements ..
      WRITE (NOUT,*) 'D03NEF Example Program Results'
      WRITE (NOUT,*)
*
*     Skip heading in data file
*
      READ (NIN,*)
*
*     Read problem parameters
*
```

```
      READ (NIN,*) KOPT
      READ (NIN,*) X
      READ (NIN,*) TMAT
      READ (NIN,*) NS, NT
      READ (NIN,*) S(1), S(NS)
      READ (NIN,*) T(1), T(NT)
      READ (NIN,*) NTD
      READ (NIN,*) (TD(I),I=1,NTD)
      READ (NIN,*) (RD(I),I=1,NTD)
      READ (NIN,*) (SIGD(I),I=1,NTD)
*
      TDPAR(1) = .TRUE.
      TDPAR(2) = .FALSE.
      TDPAR(3) = .TRUE.
      Q(1) = 0.e0
      LWORK = 9*NTD + 24
*
      IF (NS.LT.2 .OR. NS.GT.NSMAX) THEN
         WRITE (NOUT,*) 'NS invalid.'
      ELSE IF (NT.LT.2 .OR. NT.GT.NTMAX) THEN
         WRITE (NOUT,*) 'NT invalid.'
      ELSE
*
         DS = (S(NS)-S(1))/real(NS-1)
         DT = (T(NT)-T(1))/real(NT-1)
*
*        Loop over times
*
         DO 40 J = 1, NT
*
            T(J) = T(1) + (J-1)*DT
*
*           Find average values of r and sigma
*
            IFAIL = 0
            CALL D03NEF(T(J),TMAT,NTD,TD,RD,RA,WORK,LWORK,IFAIL)
            IFAIL = 0
            CALL D03NEF(T(J),TMAT,NTD,TD,SIGD,SIGA,WORK,LWORK,IFAIL)
*
*           Loop over stock prices
*
            DO 20 I = 1, NS
*
               S(I) = S(1) + (I-1)*DS
*
*              Evaluate analytic solution of Black-Scholes equation
*
               IFAIL = 0
               CALL D03NDF(KOPT,X,S(I),T(J),TMAT,TDPAR,RA,Q,SIGA,F(I,J),
     +                     THETA(I,J),DELTA(I,J),GAMMA(I,J),LAMBDA(I,J),
     +                     RHO(I,J),IFAIL)
*
   20       CONTINUE
   40    CONTINUE
*
*        Output option values.
*
         WRITE (NOUT,*)
         WRITE (NOUT,*) 'Option Values'
         WRITE (NOUT,*) '-------------'
         WRITE (NOUT,*) ' Stock Price  |   Time to Maturity (months)'
         WRITE (NOUT,99999) '|', (12*(TMAT-T(J)),J=1,NT)
         WRITE (NOUT,'(11A)') '-----------------',
     +     ('------------',J=1,NT)
         DO 60 I = 1, NS
            WRITE (NOUT,99998) S(I), '|', (F(I,J),J=1,NT)
   60    CONTINUE
*
         DO 100 IGREEK = 1, 5
*
            IF (GPRNT(IGREEK)) THEN
```

```
               WRITE (NOUT,*)
               WRITE (NOUT,*) GNAME(IGREEK)
               WRITE (NOUT,*) '------'
               WRITE (NOUT,*)
     +           ' Stock Price  |   Time to Maturity (months)'
               WRITE (NOUT,99999) '|', (12*(TMAT-T(J)),J=1,NT)
               WRITE (NOUT,'(11A)') '-----------------',
     +           ('------------',J=1,NT)
               DO 80 I = 1, NS
                  IF (IGREEK.EQ.1) THEN
                     WRITE (NOUT,99998) S(I), '|', (THETA(I,J),J=1,NT)
                  ELSE IF (IGREEK.EQ.2) THEN
                     WRITE (NOUT,99998) S(I), '|', (DELTA(I,J),J=1,NT)
                  ELSE IF (IGREEK.EQ.3) THEN
                     WRITE (NOUT,99998) S(I), '|', (GAMMA(I,J),J=1,NT)
                  ELSE IF (IGREEK.EQ.4) THEN
                     WRITE (NOUT,99998) S(I), '|', (LAMBDA(I,J),J=1,NT)
                  ELSE IF (IGREEK.EQ.5) THEN
                     WRITE (NOUT,99998) S(I), '|', (RHO(I,J),J=1,NT)
                  END IF
   80          CONTINUE
            END IF
*
  100    CONTINUE
*
      END IF
*
      STOP
*
99999 FORMAT (15X,A,1X,12(1P,e12.4))
99998 FORMAT (1P,e12.4,3X,A,1X,12(1P,e12.4))
      END
```

9.2 Program Data

```
D03NEF Example Program Data
  2                   KOPT
  50.                 X
  0.4166667           TMAT
  21  4               NS, NT
  0.0  100.           S(1), S(NS)
  0.0  0.125          T(1), T(NT)
  6                   NTD
  0.00 0.10 0.20
  0.30 0.40 0.50      TD(1) ... TD(NTD)
  0.10 0.11 0.12
  0.13 0.14 0.15      RD(1) ... RD(NTD)
  0.30 0.46 0.54
  0.54 0.46 0.30      SIGD(1) ... SIGD(NTD)
```

9.3 Program Results

```
 D03NEF Example Program Results


 Option Values
 -------------
  Stock Price  |   Time to Maturity (months)
               |   5.0000E+00  4.5000E+00  4.0000E+00  3.5000E+00
---------------------------------------------------------------
  0.0000E+00   |   0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00
  5.0000E+00   |   3.3671E-13  7.7404E-14  7.3210E-15  2.0179E-16
  1.0000E+01   |   5.2088E-07  2.4281E-07  7.2216E-08  1.1540E-08
  1.5000E+01   |   2.6607E-04  1.6753E-04  8.0943E-05  2.7179E-05
  2.0000E+01   |   8.9697E-03  6.6505E-03  4.1780E-03  2.0942E-03
  2.5000E+01   |   8.3647E-02  6.8467E-02  5.0375E-02  3.2105E-02
  3.0000E+01   |   3.8221E-01  3.3331E-01  2.7117E-01  2.0119E-01
  3.5000E+01   |   1.1298E+00  1.0275E+00  8.9292E-01  7.3146E-01
  4.0000E+01   |   2.5164E+00  2.3541E+00  2.1380E+00  1.8699E+00
  4.5000E+01   |   4.6249E+00  4.4110E+00  4.1267E+00  3.7700E+00
```

```
  5.0000E+01   |   7.4287E+00  7.1797E+00  6.8531E+00  6.4449E+00
  5.5000E+01   |   1.0830E+01  1.0564E+01  1.0221E+01  9.7996E+00
  6.0000E+01   |   1.4707E+01  1.4436E+01  1.4097E+01  1.3689E+01
  6.5000E+01   |   1.8937E+01  1.8671E+01  1.8348E+01  1.7968E+01
  7.0000E+01   |   2.3421E+01  2.3164E+01  2.2860E+01  2.2514E+01
  7.5000E+01   |   2.8080E+01  2.7833E+01  2.7550E+01  2.7234E+01
  8.0000E+01   |   3.2857E+01  3.2620E+01  3.2354E+01  3.2064E+01
  8.5000E+01   |   3.7713E+01  3.7484E+01  3.7233E+01  3.6963E+01
  9.0000E+01   |   4.2620E+01  4.2398E+01  4.2158E+01  4.1904E+01
  9.5000E+01   |   4.7561E+01  4.7344E+01  4.7112E+01  4.6868E+01
  1.0000E+02   |   5.2523E+01  5.2310E+01  5.2084E+01  5.1848E+01

 Theta
 ------
  Stock Price  |   Time to Maturity (months)
               |   5.0000E+00  4.5000E+00  4.0000E+00  3.5000E+00
---------------------------------------------------------------
  0.0000E+00   |   0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00
  5.0000E+00   |  -8.9082E-12 -3.4507E-12 -5.0884E-13 -2.1236E-14
  1.0000E+01   |  -7.2097E-06 -5.5915E-06 -2.5721E-06 -6.1830E-07
  1.5000E+01   |  -2.2499E-03 -2.3259E-03 -1.7227E-03 -8.6349E-04
  2.0000E+01   |  -4.9483E-02 -5.9355E-02 -5.6562E-02 -4.1921E-02
  2.5000E+01   |  -3.1200E-01 -4.0620E-01 -4.4765E-01 -4.1683E-01
  3.0000E+01   |  -9.8578E-01 -1.3408E+00 -1.6092E+00 -1.7186E+00
  3.5000E+01   |  -2.0479E+00 -2.8395E+00 -3.5745E+00 -4.1390E+00
  4.0000E+01   |  -3.2501E+00 -4.5165E+00 -5.8147E+00 -7.0323E+00
  4.5000E+01   |  -4.3144E+00 -5.9349E+00 -7.6762E+00 -9.4488E+00
  5.0000E+01   |  -5.0802E+00 -6.8543E+00 -8.7919E+00 -1.0815E+01
  5.5000E+01   |  -5.5225E+00 -7.2603E+00 -9.1500E+00 -1.1104E+01
  6.0000E+01   |  -5.7006E+00 -7.2722E+00 -8.9491E+00 -1.0625E+01
  6.5000E+01   |  -5.7014E+00 -7.0446E+00 -8.4366E+00 -9.7565E+00
  7.0000E+01   |  -5.6037E+00 -6.7093E+00 -7.8142E+00 -8.7951E+00
  7.5000E+01   |  -5.4653E+00 -6.3555E+00 -7.2107E+00 -7.9170E+00
  8.0000E+01   |  -5.3218E+00 -6.0329E+00 -6.6903E+00 -7.1974E+00
  8.5000E+01   |  -5.1920E+00 -5.7627E+00 -6.2736E+00 -6.6481E+00
  9.0000E+01   |  -5.0833E+00 -5.5487E+00 -5.9563E+00 -6.2492E+00
  9.5000E+01   |  -4.9969E+00 -5.3857E+00 -5.7234E+00 -5.9700E+00
  1.0000E+02   |  -4.9306E+00 -5.2651E+00 -5.5570E+00 -5.7797E+00

 Delta
 ------
  Stock Price  |   Time to Maturity (months)
               |   5.0000E+00  4.5000E+00  4.0000E+00  3.5000E+00
---------------------------------------------------------------
  0.0000E+00   |   0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00
  5.0000E+00   |   1.6086E-12  3.8832E-13  3.9572E-14  1.2111E-15
  1.0000E+01   |   8.9933E-07  4.3972E-07  1.4063E-07  2.4884E-08
  1.5000E+01   |   2.3975E-04  1.5810E-04  8.1943E-05  3.0366E-05
  2.0000E+01   |   4.9150E-03  3.8095E-03  2.5596E-03  1.4100E-03
  2.5000E+01   |   3.0345E-02  2.5906E-02  2.0311E-02  1.4153E-02
  3.0000E+01   |   9.6991E-02  8.7980E-02  7.5946E-02  6.1231E-02
  3.5000E+01   |   2.0863E-01  1.9675E-01  1.8053E-01  1.5957E-01
  4.0000E+01   |   3.4875E-01  3.3719E-01  3.2158E-01  3.0109E-01
  4.5000E+01   |   4.9361E-01  4.8480E-01  4.7356E-01  4.5924E-01
  5.0000E+01   |   6.2450E-01  6.1931E-01  6.1363E-01  6.0735E-01
  5.5000E+01   |   7.3200E-01  7.3000E-01  7.2907E-01  7.2954E-01
  6.0000E+01   |   8.1439E-01  8.1462E-01  8.1681E-01  8.2145E-01
  6.5000E+01   |   8.7440E-01  8.7589E-01  8.7961E-01  8.8602E-01
  7.0000E+01   |   9.1650E-01  9.1850E-01  9.2260E-01  9.2911E-01
  7.5000E+01   |   9.4522E-01  9.4726E-01  9.5107E-01  9.5679E-01
  8.0000E+01   |   9.6441E-01  9.6624E-01  9.6946E-01  9.7406E-01
  8.5000E+01   |   9.7704E-01  9.7856E-01  9.8111E-01  9.8461E-01
  9.0000E+01   |   9.8526E-01  9.8646E-01  9.8839E-01  9.9094E-01
  9.5000E+01   |   9.9057E-01  9.9148E-01  9.9290E-01  9.9470E-01
  1.0000E+02   |   9.9397E-01  9.9464E-01  9.9567E-01  9.9691E-01

 Gamma
 ------
  Stock Price  |   Time to Maturity (months)
               |   5.0000E+00  4.5000E+00  4.0000E+00  3.5000E+00
---------------------------------------------------------------
```

0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
5.0000E+00	7.2334E-12	1.8390E-12	2.0276E-13	6.9267E-15
1.0000E+01	1.4139E-06	7.2829E-07	2.5205E-07	4.9786E-08
1.5000E+01	1.8932E-04	1.3153E-04	7.3756E-05	3.0494E-05
2.0000E+01	2.2528E-03	1.8392E-03	1.3360E-03	8.2017E-04
2.5000E+01	8.6933E-03	7.8126E-03	6.6135E-03	5.1251E-03
3.0000E+01	1.8099E-02	1.7264E-02	1.6056E-02	1.4350E-02
3.5000E+01	2.5953E-02	2.5691E-02	2.5315E-02	2.4683E-02
4.0000E+01	2.9260E-02	2.9618E-02	3.0194E-02	3.0968E-02
4.5000E+01	2.8046E-02	2.8736E-02	2.9814E-02	3.1368E-02
5.0000E+01	2.4005E-02	2.4715E-02	2.5793E-02	2.7346E-02
5.5000E+01	1.8950E-02	1.9500E-02	2.0296E-02	2.1401E-02
6.0000E+01	1.4105E-02	1.4449E-02	1.4903E-02	1.5476E-02
6.5000E+01	1.0054E-02	1.0221E-02	1.0396E-02	1.0555E-02
7.0000E+01	6.9401E-03	6.9861E-03	6.9806E-03	6.8890E-03
7.5000E+01	4.6779E-03	4.6538E-03	4.5552E-03	4.3505E-03
8.0000E+01	3.0978E-03	3.0414E-03	2.9096E-03	2.6800E-03
8.5000E+01	2.0250E-03	1.9598E-03	1.8291E-03	1.6205E-03
9.0000E+01	1.3114E-03	1.2499E-03	1.1365E-03	9.6637E-04
9.5000E+01	8.4362E-04	7.9138E-04	7.0024E-04	5.7052E-04
1.0000E+02	5.4033E-04	4.9856E-04	4.2893E-04	3.3442E-04

Lambda

Stock Price	Time to Maturity (months)			
	5.0000E+00	4.5000E+00	4.0000E+00	3.5000E+00
0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
5.0000E+00	3.6558E-11	8.6441E-12	8.6672E-13	2.6259E-14
1.0000E+01	2.8583E-05	1.3693E-05	4.3098E-06	7.5495E-07
1.5000E+01	8.6115E-03	5.5645E-03	2.8375E-03	1.0404E-03
2.0000E+01	1.8217E-01	1.3832E-01	9.1376E-02	4.9748E-02
2.5000E+01	1.0984E+00	9.1808E-01	7.0676E-01	4.8574E-01
3.0000E+01	3.2931E+00	2.9214E+00	2.4708E+00	1.9584E+00
3.5000E+01	6.4272E+00	5.9173E+00	5.3025E+00	4.5851E+00
4.0000E+01	9.4643E+00	8.9101E+00	8.2604E+00	7.5135E+00
4.5000E+01	1.1481E+01	1.0941E+01	1.0323E+01	9.6323E+00
5.0000E+01	1.2132E+01	1.1617E+01	1.1026E+01	1.0367E+01
5.5000E+01	1.1588E+01	1.1091E+01	1.0498E+01	9.8169E+00
6.0000E+01	1.0265E+01	9.7801E+00	9.1734E+00	8.4486E+00
6.5000E+01	8.5872E+00	8.1198E+00	7.5104E+00	6.7621E+00
7.0000E+01	6.8747E+00	6.4363E+00	5.8487E+00	5.1188E+00
7.5000E+01	5.3194E+00	4.9219E+00	4.3812E+00	3.7109E+00
8.0000E+01	4.0081E+00	3.6599E+00	3.1840E+00	2.6009E+00
8.5000E+01	2.9578E+00	2.6623E+00	2.2597E+00	1.7754E+00
9.0000E+01	2.1474E+00	1.9036E+00	1.5741E+00	1.1870E+00
9.5000E+01	1.5392E+00	1.3429E+00	1.0806E+00	7.8078E-01
1.0000E+02	1.0923E+00	9.3740E-01	7.3341E-01	5.0711E-01

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Stock Price	Time to Maturity (months)			
	5.0000E+00	4.5000E+00	4.0000E+00	3.5000E+00
0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
5.0000E+00	3.2110E-12	6.9908E-13	6.3513E-14	1.7073E-15
1.0000E+01	3.5302E-06	1.5579E-06	4.4470E-07	6.9214E-08
1.5000E+01	1.3876E-03	8.2648E-04	3.8273E-04	1.2492E-04
2.0000E+01	3.7221E-02	2.6077E-02	1.5671E-02	7.6142E-03
2.5000E+01	2.8124E-01	2.1719E-01	1.5247E-01	9.3836E-02
3.0000E+01	1.0531E+00	8.6478E-01	6.6907E-01	4.7709E-01
3.5000E+01	2.5718E+00	2.1971E+00	1.8086E+00	1.4156E+00
4.0000E+01	4.7641E+00	4.1750E+00	3.5750E+00	2.9673E+00
4.5000E+01	7.3281E+00	6.5270E+00	5.7279E+00	4.9280E+00
5.0000E+01	9.9152E+00	8.9196E+00	7.9427E+00	6.9774E+00
5.5000E+01	1.2262E+01	1.1095E+01	9.9592E+00	8.8448E+00
6.0000E+01	1.4232E+01	1.2915E+01	1.1637E+01	1.0383E+01
6.5000E+01	1.5791E+01	1.4348E+01	1.2942E+01	1.1557E+01
7.0000E+01	1.6973E+01	1.5424E+01	1.3907E+01	1.2403E+01
7.5000E+01	1.7838E+01	1.6204E+01	1.4594E+01	1.2987E+01
8.0000E+01	1.8457E+01	1.6755E+01	1.5067E+01	1.3376E+01

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 8.5000E+01   |   1.8890E+01  1.7135E+01  1.5387E+01  1.3629E+01
 9.0000E+01   |   1.9189E+01  1.7393E+01  1.5599E+01  1.3790E+01
 9.5000E+01   |   1.9393E+01  1.7567E+01  1.5738E+01  1.3891E+01
 1.0000E+02   |   1.9531E+01  1.7683E+01  1.5827E+01  1.3954E+01
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