

NAG Fortran Library Routine Document

D01BAF

Note: before using this routine, please read the Users' Note for your implementation to check the interpretation of ***bold italicised*** terms and other implementation-dependent details.

1 Purpose

D01BAF computes an estimate of the definite integral of a function of known analytical form, using a Gaussian quadrature formula with a specified number of abscissae. Formulae are provided for a finite interval (Gauss–Legendre), a semi-infinite interval (Gauss–Laguerre, Gauss–Rational), and an infinite interval (Gauss–Hermite).

2 Specification

```
real FUNCTION D01BAF(D01XXX, A, B, N, FUN, IFAIL)
  INTEGER          N, IFAIL
  real             A, B, FUN
  EXTERNAL         D01XXX, FUN
```

3 Description

3.1 General

This routine evaluates an estimate of the definite integral of a function $f(x)$, over a finite or infinite range, by n -point Gaussian quadrature (see Davis and Rabinowitz (1975), Fröberg (1970), Ralston (1965) or Stroud and Secrest (1966)). The integral is approximated by a summation

$$\sum_{i=1}^n w_i f(x_i)$$

where the w_i are called the weights, and the x_i the abscissae. A selection of values of n is available. (See Section 5.)

3.2 Both Limits Finite

$$\int_a^b f(x) dx.$$

The Gauss–Legendre weights and abscissae are used, and the formula is exact for any function of the form:

$$f(x) = \sum_{i=0}^{2n-1} c_i x^i.$$

The formula is appropriate for functions which can be well approximated by such a polynomial over $[a, b]$. It is inappropriate for functions with algebraic singularities at one or both ends of the interval, such as $(1+x)^{-1/2}$ on $[-1, 1]$.

3.3 One Limit Infinite

$$\int_a^\infty f(x) dx \quad \text{or} \quad \int_{-\infty}^a f(x) dx.$$

Two quadrature formulae are available for these integrals.

- (a) The Gauss–Laguerre formula is exact for any function of the form:

$$f(x) = e^{-bx} \sum_{i=0}^{2n-1} c_i x^i.$$

This formula is appropriate for functions decaying exponentially at infinity; the parameter b should be chosen if possible to match the decay rate of the function.

- (b) The Gauss–Rational formula is exact for any function of the form:

$$f(x) = \sum_{i=2}^{2n+1} \frac{c_i}{(x+b)^i} = \frac{\sum_{i=0}^{2n-1} c_{2n+1-i} (x+b)^i}{(x+b)^{2n+1}}.$$

This formula is likely to be more accurate for functions having only an inverse power rate of decay for large x . Here the choice of a suitable value of b may be more difficult; unfortunately a poor choice of b can make a large difference to the accuracy of the computed integral.

3.4 Both Limits Infinite

$$\int_{-\infty}^{+\infty} f(x) dx.$$

The Gauss–Hermite weights and abscissae are used, and the formula is exact for any function of the form:

$$f(x) = e^{-b(x-a)^2} \sum_{i=0}^{2n-1} c_i x^i.$$

Again, for general functions not of this exact form, the parameter b should be chosen to match if possible the decay rate at $\pm\infty$.

4 References

Davis P J and Rabinowitz P (1975) *Methods of Numerical Integration* Academic Press

Fröberg C E (1970) *Introduction to Numerical Analysis* Addison-Wesley

Ralston A (1965) *A First Course in Numerical Analysis* pp. 87–90 McGraw-Hill

Stroud A H and Secrest D (1966) *Gaussian Quadrature Formulas* Prentice-Hall

5 Parameters

- 1: D01XXX – SUBROUTINE, supplied by the NAG Fortran Library. *External Procedure*

The name of the routine indicates the quadrature formula:

D01BAZ, for Gauss–Legendre quadrature on a finite interval;

D01BAY, for Gauss–Rational quadrature on a semi-infinite interval;

D01BAX, for Gauss–Laguerre quadrature on a semi-infinite interval;

D01BAW, for Gauss–Hermite quadrature on an infinite interval.

The name used must be declared as EXTERNAL in the (sub)program from which D01BAF is called.

- 2: A – *real* *Input*
 3: B – *real* *Input*

On entry: the parameters a and b which occur in the integration formulae:

Gauss–Legendre:

a is the lower limit and b is the upper limit of the integral. It is not necessary that $a < b$.

Gauss–Rational:

b must be chosen so as to make the integrand match as closely as possible the exact form given in Section 3.3(b). The range of integration is $[a, \infty)$ if $a + b > 0$, and $(-\infty, a]$ if $a + b < 0$.

Gauss–Laguerre:

b must be chosen so as to make the integrand match as closely as possible the exact form given in Section 3.3(a). The range of integration is $[a, \infty)$ if $b > 0$, and $(-\infty, a]$ if $b < 0$.

Gauss–Hermite:

a and b must be chosen so as to make the integrand match as closely as possible the exact form given in Section 3.4.

Constraints:

Gauss–Rational: $A + B \neq 0$,

Gauss–Laguerre: $B \neq 0$,

Gauss–Hermite: $B > 0$.

4: N – INTEGER

Input

On entry: the number of abscissae to be used, n .

Constraint: $N = 1, 2, 3, 4, 5, 6, 8, 10, 12, 14, 16, 20, 24, 32, 48$ or 64 .

5: FUN – **real** FUNCTION, supplied by the user.

External Procedure

FUN must return the value of the integrand f at a given point.

Its specification is:

<pre> real FUNCTION FUN(X) real X 1: X – real On entry: the point at which the integrand must be evaluated. </pre>	
	<i>Input</i>

FUN must be declared as EXTERNAL in the (sub)program from which D01BAF is called. Parameters denoted as *Input* must **not** be changed by this procedure.

Some points to bear in mind when coding FUN are mentioned in Section 7.

6: IFAIL – INTEGER

Input/Output

On entry: IFAIL must be set to 0, -1 or 1 . Users who are unfamiliar with this parameter should refer to Chapter P01 for details.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

For environments where it might be inappropriate to halt program execution when an error is detected, the value -1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, because for this routine the values of the output parameters may be useful even if IFAIL $\neq 0$ on exit, the recommended value is -1 . **When the value -1 or 1 is used it is essential to test the value of IFAIL on exit.**

6 Error Indicators and Warnings

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

IFAIL = 1

The N-point rule is not among those stored. If the soft fail option is used, the answer is evaluated for the largest valid value of N less than the requested value.

IFAIL = 2

The value of A and/or B is invalid.

Gauss-Rational: $A + B = 0$.

Gauss-Laguerre: $B = 0$.

Gauss-Hermite: $B \leq 0$.

If the soft fail option is used, the answer is returned as zero.

7 Accuracy

The accuracy depends on the behaviour of the integrand, and on the number of abscissae used. No tests are carried out in the routine to estimate the accuracy of the result. If such an estimate is required, the routine may be called more than once, with a different number of abscissae each time, and the answers compared. It is to be expected that for sufficiently smooth functions a larger number of abscissae will give improved accuracy.

Alternatively, the range of integration may be subdivided, the integral estimated separately for each sub-interval, and the sum of these estimates compared with the estimate over the whole range.

The coding of the function FUN may also have a bearing on the accuracy. For example, if a high-order Gauss-Laguerre formula is used, and the integrand is of the form

$$f(x) = e^{-bx} g(x)$$

it is possible that the exponential term may underflow for some large abscissae. Depending on the machine, this may produce an error, or simply be assumed to be zero. In any case, it would be better to evaluate the expression as:

$$f(x) = \exp(-bx + \ln g(x))$$

Another situation requiring care is exemplified by

$$\int_{-\infty}^{+\infty} e^{-x^2} x^m dx = 0, \quad m \text{ odd.}$$

The integrand here assumes very large values; for example, for $m = 63$, the peak value exceeds 3×10^{33} . Now, if the machine holds floating-point numbers to an accuracy of k significant decimal digits, we could not expect such terms to cancel in the summation leaving an answer of much less than 10^{33-k} (the weights being of order unity); that is instead of zero, we obtain a rather large answer through rounding error. Fortunately, such situations are characterised by great variability in the answers returned by formulae with different values of n . In general, the user should be aware of the order of magnitude of the integrand, and should judge the answer in that light.

8 Further Comments

The time taken by the routine depends on the complexity of the expression for the integrand and on the number of abscissae required.

9 Example

This example program evaluates the integrals

$$\int_0^1 \frac{4}{1+x^2} dx = \pi$$

by Gauss–Legendre quadrature;

$$\int_2^\infty \frac{1}{x^2 \ln x} dx = 0.378671$$

by Gauss–Rational quadrature with $b = 0$;

$$\int_2^\infty \frac{e^{-x}}{x} dx = 0.048901$$

by Gauss–Laguerre quadrature with $b = 1$; and

$$\int_{-\infty}^{+\infty} e^{-3x^2-4x-1} dx = \int_{-\infty}^{+\infty} e^{-3(x+1)^2} e^{2x+2} dx = 1.428167$$

by Gauss–Hermite quadrature with $a = -1$ and $b = 3$.

The formulae with $n = 4, 8, 16$ are used in each case.

9.1 Program Text

Note: the listing of the example program presented below uses ***bold italicised*** terms to denote precision-dependent details. Please read the Users’ Note for your implementation to check the interpretation of these terms. As explained in the Essential Introduction to this manual, the results produced may not be identical for all implementations.

```
*      D01BAF Example Program Text
*      Mark 14 Revised.  NAG Copyright 1989.
*      .. Parameters ..
      INTEGER          NOUT
      PARAMETER        (NOUT=6)
*      .. Local Scalars ..
      real              A, ANS, B
      INTEGER           I, IFAIL
*      .. Local Arrays ..
      INTEGER           NSTOR(3)
*      .. External Functions ..
      real              D01BAF, FUN1, FUN2, FUN3, FUN4
      EXTERNAL          D01BAF, FUN1, FUN2, FUN3, FUN4
*      .. External Subroutines ..
      EXTERNAL          D01BAW, D01BAX, D01BAY, D01BAZ
*      .. Data statements ..
      DATA             NSTOR/4, 8, 16/
*      .. Executable Statements ..
      WRITE (NOUT,*) 'D01BAF Example Program Results'
      WRITE (NOUT,*)
      WRITE (NOUT,*) 'Gauss-Legendre example'
      DO 20 I = 1, 3
        A = 0.0e0
        B = 1.0e0
        IFAIL = 1

*
        ANS = D01BAF(D01BAZ,A,B,NSTOR(I),FUN1,IFAIL)
*
        IF (IFAIL.NE.0) THEN
          WRITE (NOUT,99998) 'IFAIL = ', IFAIL
          WRITE (NOUT,*)
        END IF
        IF (IFAIL.LE.1) WRITE (NOUT,99999) NSTOR(I),
+          ' Points      Answer = ', ANS
20 CONTINUE
      WRITE (NOUT,*)
      WRITE (NOUT,*)
      WRITE (NOUT,*) 'Gauss-Rational example'
```

```

DO 40 I = 1, 3
  A = 2.0e0
  B = 0.0e0
  IFAIL = 1
*
  ANS = D01BAF(D01BAY,A,B,NSTOR(I),FUN2,IFAIL)
*
  IF (IFAIL.NE.0) THEN
    WRITE (NOUT,99998) 'IFAIL = ', IFAIL
    WRITE (NOUT,*)
  END IF
  IF (IFAIL.LE.1) WRITE (NOUT,99999) NSTOR(I),
+    ' Points      Answer = ', ANS
40 CONTINUE
  WRITE (NOUT,*)
  WRITE (NOUT,*)
  WRITE (NOUT,*) 'Gauss-Laguerre example'
  DO 60 I = 1, 3
    IFAIL = 1
    A = 2.0e0
    B = 1.0e0
*
    ANS = D01BAF(D01BAX,A,B,NSTOR(I),FUN3,IFAIL)
*
    IF (IFAIL.NE.0) THEN
      WRITE (NOUT,99998) 'IFAIL = ', IFAIL
      WRITE (NOUT,*)
    END IF
    IF (IFAIL.LE.1) WRITE (NOUT,99999) NSTOR(I),
+      ' Points      Answer = ', ANS
60 CONTINUE
  WRITE (NOUT,*)
  WRITE (NOUT,*)
  WRITE (NOUT,*) 'Gauss-Hermite example'
  DO 80 I = 1, 3
    A = -1.0e0
    B = 3.0e0
    IFAIL = 1
*
    ANS = D01BAF(D01BAW,A,B,NSTOR(I),FUN4,IFAIL)
*
    IF (IFAIL.NE.0) THEN
      WRITE (NOUT,99998) 'IFAIL = ', IFAIL
      WRITE (NOUT,*)
    END IF
    IF (IFAIL.LE.1) WRITE (NOUT,99999) NSTOR(I),
+      ' Points      Answer = ', ANS
80 CONTINUE
  STOP
*
99999 FORMAT (1X,I5,A,F10.5)
99998 FORMAT (1X,A,I2)
END
*
real FUNCTION FUN1(X)
* .. Scalar Arguments ..
real X
* .. Executable Statements ..
FUN1 = 4.0e0/(1.0e0+X*X)
RETURN
END
*
real FUNCTION FUN2(X)
* .. Scalar Arguments ..
real X
* .. Intrinsic Functions ..
INTRINSIC LOG
* .. Executable Statements ..
FUN2 = 1.0e0/(X*X*LOG(X))
RETURN
END

```

```

*
  real FUNCTION FUN3(X)
*
  .. Scalar Arguments ..
  real X
*
  .. Intrinsic Functions ..
  INTRINSIC EXP
*
  .. Executable Statements ..
  FUN3 = EXP(-X)/X
  RETURN
  END

*
  real FUNCTION FUN4(X)
*
  .. Scalar Arguments ..
  real X
*
  .. Intrinsic Functions ..
  INTRINSIC EXP
*
  .. Executable Statements ..
  FUN4 = EXP(-3.0e0*X*X-4.0e0*X-1.0e0)
  RETURN
  END

```

9.2 Program Data

None.

9.3 Program Results

D01BAF Example Program Results

Gauss-Legendre example

4 Points	Answer =	3.14161
8 Points	Answer =	3.14159
16 Points	Answer =	3.14159

Gauss-Rational example

4 Points	Answer =	0.37910
8 Points	Answer =	0.37876
16 Points	Answer =	0.37869

Gauss-Laguerre example

4 Points	Answer =	0.04887
8 Points	Answer =	0.04890
16 Points	Answer =	0.04890

Gauss-Hermite example

4 Points	Answer =	1.42803
8 Points	Answer =	1.42817
16 Points	Answer =	1.42817
