COMSOL MULTIPHYSICS®

MODELING GUIDE

VERSION 3.4

COMSOL

How to contact COMSOL:

Benelux
COMSOL BV
Röntgenlaan 19
2719 DX Zoetermeer
The Netherlands
Phone: +31 (0) 79 363 4230
Fax: +31 (0) 79 361 4212
info@femlab.nl
www.femlab.nl

Denmark
COMSOL A/S
Diplomvej 376
2800 Kgs. Lyngby
Phone: +45 88 70 82 00
Fax: +45 88 70 80 90
info@comsol.dk
www.comsol.dk

Finland
COMSOL OY
Arabianranta 6
FIN-00560 Helsinki
Phone: +358 9 2510 400
Fax: +358 9 2510 4010
info@comsol.fi
www.comsol.fi

France
COMSOL France
WTC, 5 pl. Robert Schuman
F-38000 Grenoble
Phone: +33 (0)4 76 46 49 01
Fax: +33 (0)4 76 46 07 42
info@comsol.fr
www.comsol.fr

Germany
FEMLAB GmbH
Berliner Str. 4
D-37073 Göttingen
Phone: +49-551-99721-0
Fax: +49-551-99721-29
info@femlab.de
www.femlab.de

Italy
COMSOL S.r.l.
Via Vittorio Emanuele II, 22
25122 Brescia
Phone: +39-030-3793800
Fax: +39-030-3793899
info.it@comsol.com
www.it.comsol.com

Norway
COMSOL AS
Søndre gate 7
NO-7485 Trondheim
Phone: +47 73 84 24 00
Fax: +47 73 84 24 01
info@comsol.no
www.comsol.no

Sweden
COMSOL AB
Tegnérgatan 23
SE-111 40 Stockholm
Phone: +46 8 412 95 00
Fax: +46 8 412 95 10
info@comsol.se
www.comsol.se

Switzerland
FEMLAB GmbH
Technoparkstrasse 1
CH-8005 Zürich
Phone: +41 (0)44 445 2140
Fax: +41 (0)44 445 2141
info@femlab.ch
www.femlab.ch

United Kingdom
COMSOL Ltd.
UH Innovation Centre
College Lane
Hatfield
Hertfordshire AL10 9AB
Phone:+44-(0)-1707 284747
Fax: +44-(0)-1707 284746
info.uk@comsol.com
www.uk.comsol.com

United States
COMSOL, Inc.
1 New England Executive Park
Suite 350
Burlington, MA 01803
Phone: +1-781-273-3322
Fax: +1-781-273-6603

COMSOL, Inc.
10850 Wilshire Boulevard
Suite 800
Los Angeles, CA 90024
Phone: +1-310-441-4800
Fax: +1-310-441-0868

COMSOL, Inc.
744 Cowper Street
Palo Alto, CA 94301
Phone: +1-650-324-9935
Fax: +1-650-324-9936

info@comsol.com
www.comsol.com

For a complete list of international representatives, visit
www.comsol.com/contact

Company home page
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COMSOL user forums
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COMSOL Multiphysics Modeling Guide

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Introduction

This *COMSOL Multiphysics Modeling Guide* provides an in-depth examination of the application modes in COMSOL Multiphysics and how to use them to model different types of physics and to perform equation-based modeling using PDEs.

This chapter gives an overview of the available application modes as well as some general guidelines for effective modeling.

Overview of the COMSOL Multiphysics Application Modes

Solving PDEs generally means you must take the time to set up the underlying equations, material properties, and boundary conditions for a given problem. COMSOL Multiphysics, however, relieves you of much of this work. The package provides a number of *application modes* that consist of predefined templates and user interfaces already set up with equations and variables for specific areas of physics. Special properties allow the selection of, for instance, analysis type and model formulations. The application mode interfaces consist of customized dialog boxes for the physics in subdomains and on boundaries, edges, and points along with predefined PDEs. A set of application-dependent variables makes it easy to visualize and postprocess the important physical quantities using conventional terminology and notation. Adding even more flexibility, the *equation system view* provides the possibility to examine and modify the underlying PDEs in the case where a predefined application mode does not exactly match the application you want to model.

Note: Suites of application modes that are optimized for specific disciplines together with large model libraries are available in a group of optional products: the *AC/DC Module,* the *Acoustics Module*, the *Chemical Engineering Module*, the *Earth Science Module*, the *Heat Transfer Module*, the *MEMS Module*, the *RF Module*, and the *Structural Mechanics Module*.

Application Modes in COMSOL Multiphysics

The following table lists the application modes in COMSOL Multiphysics and their availability for 1D, 1D axisymmetric, 2D, 2D axisymmetric, and 3D geometries:

APPLICATION MODE	1D	1D AXI	2D	2D AXI	3D
Acoustics					
Acoustics			√	√	√
Diffusion					
Convection and Diffusion	√	√	√	√	√

APPLICATION MODE	1D	1D AXI	2D	2D AXI	3D
Diffusion	√	√	√	√	√
Electromagnetics					
AC Power Electromagnetics			√		
Conductive Media DC			√	√	√
Electrostatics			√	√	√
Magnetostatics			√	√	√
Heat Transfer					
Convection and Conduction	√	√	√	√	√
Conduction	√	√	√	√	√
Fluid Dynamics					
Incompressible Navier-Stokes			√	√	√
Structural Mechanics					
Plane Strain			√		
Plane Stress			√		
Axial Symmetry, Stress-Strain				√	
Solid, Stress-Strain					√
Deformed Mesh					
Moving Mesh (ALE)	√	√	√	√	√
Parameterized Geometry			√		
PDE Modes					
Coefficient Form	√		√		√
General Form	√		√		√
Weak Form, Subdomain	√		√		√
Weak Form, Boundary	√		√		√
Weak Form, Edge					√
Weak Form, Point			√		√
Classical PDEs					
Convection-diffusion Equation	√		√		√
Laplace's Equation	√		√		√
Heat Equation	√		√		√
Helmholtz Equation	√		√		√
Poisson's Equation	√		√		√

APPLICATION MODE	1D	1D AXI	2D	2D AXI	3D
Schrödinger Equation	√		√		√
Wave Equation	√		√		√

As the table indicates, these application modes fall into four broad categories:

THE PHYSICS MODES

Use the physics modes to instantly access convenient templates for specific application areas. Here you can specify physical properties for models in fields such as structural mechanics, heat transfer, and electromagnetics. Details on how to use the physics modes appear in the section "Using the Physics Modes" on page 13.

THE DEFORMED MESH APPLICATION MODES

These application modes provide support for applications with moving boundaries using the Moving Mesh (ALE) application mode and for parameterized geometries in 2D. See the section "Deformed Meshes" on page 391 for more information.

THE PDE MODES

Turn to these modes to model directly with PDEs when you cannot find a suitable physics mode. With these modes you define the problem in terms of mathematical expressions and coefficients.

COMSOL Multiphysics includes three PDE modes:

- The *Coefficient form* allows you to solve linear or almost linear problems using PDEs and coefficients that often correspond directly to various physical properties.
- The *General form* provides a computational framework specialized for highly nonlinear problems. Consider using a weak form for these problems, too.
- The *Weak form* makes it possible to model a wider class of problems, for example models with mixed time and space derivatives, or models with phenomena on boundaries, edges, or points as described with PDEs. In terms of convergence rate, these modes also set a computational framework suited for all types of nonlinear problems.

For details on how to apply the PDE modes, see the sections "PDE Modes for Equation-Based Modeling" on page 237 and "The Weak Form" on page 291.

CLASSICAL PDES

The Classical PDEs folder contain application modes that describe a suite of well-known PDEs. They are special cases of the coefficient form PDE and are not meant to serve as templates.

Selecting an Application Mode

When creating a model in COMSOL Multiphysics, you can select a single application mode that describes one type of physics or select several application modes for multiphysics modeling and coupled-field analyses.

MODELING USING A SINGLE APPLICATION MODE

Most of the physics application modes contain stationary, eigenvalue, and dynamic (time-dependent) analysis types. As already mentioned, these modes provide a modeling interface where you can create models using material properties, boundary conditions, and initial conditions. Each of these modes comes with a template that automatically supplies the appropriate underlying PDE.

If you cannot find a physics mode that matches a given problem, try one of the PDE modes, which allow you to define a custom model in general mathematical terms. Indeed, COMSOL Multiphysics can model virtually any scientific phenomenon or engineering problem that originates from the laws of science.

MULTIPHYSICS MODELING USING MULTIPLE APPLICATION MODES

When modeling real-world systems, you often need to include the interaction between different kinds of physics. For instance, the properties of an electronic component such as an inductor vary with temperature. To solve such a problem, combine two or several application modes into a single model using the multiphysics features of COMSOL Multiphysics. For the example just mentioned, combine the Conductive Media DC and Heat Transfer by Conduction application modes. In this way you create a system of two PDEs with two dependent variables: V for the electric potential and T for the temperature. There are also predefined multiphysics couplings that provide two coupled application modes for some common multiphysics applications.

Combining physics modes and PDE modes also works well. Assume, for instance, that you want to model the fluid-structure interactions due to the vibrations of yoghurt in a cardboard container as it rides on a conveyor belt. You could start with the Plane Stress mode for structural mechanics to model the container walls and then add a PDE to model the irrotational flow of the fluid. This approach also creates a system of two

PDEs but requires that you define one of them from scratch, in this case using the general PDE formulation.

To summarize the proposed strategy for modeling processes that involve several types of physics: Look for application modes suitable for the phenomena of interest. If you find them among the physics modes, use them; if not, add one or more PDE modes.

The approach when coupling multiple application modes is to use the dependent variables or their derivatives directly, or to use expressions containing the dependent variables. The coupling can occur in subdomains and on boundaries.

Modeling Guidelines

To allow you to model large-scale problems, COMSOL Multiphysics lets you tune solver settings and use symmetries and other model properties to reach a solution or—failing that—interrupt the solution process to retrieve a partial solution.

Using Symmetries

By using symmetries in a model you can reduce its size by one-half or more, making this an efficient tool for solving large problems. This applies to the cases where the geometries and modeling assumptions include symmetries.

The most important types of symmetries are axial symmetry and symmetry and antisymmetry planes or lines:

- *Axial Symmetry* is common for cylindrical and similar 3D geometries. If the geometry is axisymmetric, there are variations in the radial (r) and vertical (z) direction only and not in the angular (θ) direction. You can then solve a 2D problem in the rz-plane instead of the full 3D model, which can save considerable memory and computation time. Many COMSOL Multiphysics application modes are available in axisymmetric versions.
- *Symmetry and Antisymmetry Planes or Lines* are common in both 2D and 3D models. *Symmetry* means that a model is identical on either side of a dividing line or plane. For a scalar field, the normal flux is zero across the symmetry line. In structural mechanics, the symmetry conditions are different. *Antisymmetry* means that the loading of a model is oppositely balanced on either side of a dividing line or plane. For a scalar field, the dependent variable is 0 along the antisymmetry plane or line. Structural mechanics applications have other antisymmetry conditions. Many application modes have symmetry conditions directly available in the **Boundary Settings** dialog box.

To take advantage of symmetry planes and symmetry lines, all of the geometry, material properties, and boundary conditions must be symmetric, and any loads or sources must be symmetric or antisymmetric. You can then build a model of the symmetric portion, which can be half, a quarter, or an eighth of the full geometry, and apply the appropriate symmetry (or antisymmetry) boundary conditions.

Effective Memory Management

Especially in 3D modeling, extensive memory usage dictates some extra precautions. First, check that you have selected an iterative linear system solver. Normally you do not need to worry about which solver to use, because the application mode makes an appropriate default choice. In some situations, though, it might be necessary to make additional changes to the solver settings and the model.

ESTIMATING THE MEMORY USE FOR A MODEL

Out-of-memory messages occur when COMSOL Multiphysics tries to allocate an array that does not fit sequentially in memory. It is common that the amount of available memory seems large enough for an array, but there might not be a contiguous block of that size due to memory fragmentation. The only solver that requires a single large contiguous memory block is the UMFPACK direct solver.

In estimating how much memory it takes to solve a specific model, the following factors are the most important:

- The number of node points
- The number of dependent and independent variables
- The element order
- The sparsity pattern of the system matrices. The sparsity pattern, in turn, depends on the shape of the geometry and the mesh. For example, an extended ellipsoid gives sparser matrices than a sphere.

CREATING A MEMORY-EFFICIENT GEOMETRY

A first step when dealing with large models is to try to reduce the model geometry as much as possible. Often you can find symmetry planes and reduce the model to a half, a quarter or even an eighth of the original size. Memory usage does not scale linearly but rather polynomially (Cn^k, $k > 1$), which means that the model needs less than half the memory if you find a symmetry plane and cut the geometry size by half. Other ways to create a more memory-efficient geometry include:

- Avoiding small geometry objects where not needed and using Bézier curves instead of polygon chains.

- Making sure that the mesh elements are of a high quality. Mesh quality is important for an iterative linear system solver. Convergence is faster and more robust if the element quality is high.
- Avoiding geometries with sharp, narrow corners. Mesh elements get thin when they approach sharp corners, leading to poor element quality in the adjacent regions.

Selecting an Element Type

As the default element type for most application modes, COMSOL Multiphysics uses second-order Lagrange elements or shape functions (see "Finite Elements" on page 452 in the *COMSOl Multiphysics Reference Guide* for an overview of the available element types). These and other higher-order elements add additional degrees of freedom on midpoint and interior nodes in the mesh elements. These added degrees of freedom provide a more accurate solution but also require more memory due to the reduced sparsity of the discretized system. For many application areas, such as stress analysis in structural and solid mechanics, the increased accuracy of a second-order element is important. In fluid-flow modeling using the incompressible Navier-Stokes equations, a combination of element types using an element for the velocity components of a higher order than that for the pressure usually provides the best result. The default element for the Incompressible Navier-Stokes application mode is the P2-P1 element using second-order elements for the velocity components and linear elements for the pressure. For other applications you can select a first-order element instead of a second-order element, or reduce the element order in general, to reduce memory use.

Analyzing Model Convergence and Accuracy

It is important that the finite element model accurately captures local variations in the solution such as stress concentrations. In some cases you can compare your results to values from handbooks, measurements, or other sources of data. Many examples in the *COMSOL Multiphysics Modeling Guide* and the *COMSOL Multiphysics Model Library* include comparisons to established results or analytical solutions. Look for these *benchmark models* as a means of checking results.

If a model has not been verified by other means, a convergence test is useful for determining if the mesh density is sufficient. Here you refine the mesh and run the analysis again, and then you see if the solution is converging to a stable value as the mesh is refined. If the solution changes when you refine the mesh, the solution is mesh dependent, so the model requires a finer mesh. You can use adaptive mesh refinement (see “Avoiding Inverted Mesh Elements” on page 356 of the *COMSOL Multiphysics User’s Guide*), which adds mesh elements based on an error criterion to resolve those areas where the error is large.

For convergence, it is important to avoid singularities in the geometry (see “Avoiding Singularities and Degeneracies in the Geometry” on page 94 of the *COMSOL Multiphysics User’s Guide* for more information).

Achieving Convergence When Solving Nonlinear Equations

Nonlinear problems are often difficult to solve. In many cases, no unique solution exists. COMSOL Multiphysics uses a Newton-type iterative method to solve nonlinear systems of PDEs. This solution method can be sensitive to the initial estimate of the solution. If the initial conditions are too far from the desired solution, convergence might be impossible, even though it might be simple from a different starting value.

You can do several things to improve the chances for finding the relevant solutions to difficult nonlinear problems:

- Provide the best possible initial values.
- Solve sequentially and iterate between single-physics equations; finish by solving the fully coupled multiphysics problem when you have obtained better starting guesses.
- Ensure that the boundary conditions are consistent with the initial solution and that neighboring boundaries have compatible conditions that do not create singularities.
- Refine the mesh in regions of steep gradients.
- For convection-type problem, introduce artificial diffusion to improve the problem’s numerical properties (see “Stabilization Techniques” on page 433).
- Scaling can be an issue when one solution component is zero. In those cases, the automatic scaling might not work. See “Scaling of Variables and Equations” on page 497 of the *COMSOL Multiphysics Reference Guide* for more information.
- Turn a stationary nonlinear PDE into a time-dependent problem. Making the problem time dependent generally results in smoother convergence. By making sure to solve the time-dependent problem for a time span long enough for the solution to reach a steady state, you solve the original stationary problem.
- Use the parametric solver and vary a material property or a PDE coefficient starting from a value that makes the equations less nonlinear to the value at which you want to compute the solution. This way you solve a series of increasingly difficult

nonlinear problems. The solution of a slightly nonlinear problem that is easy to solve serves as the initial value for a more difficult nonlinear problem.

Avoiding Strong Transients

If you start solving a time-dependent problem with initial conditions that are inconsistent, or if you use boundary settings or subdomain settings that switch instantaneously at a certain time, you induce strong transient signals in a system. The time-stepping algorithm then takes very small steps to resolve the transient, and the solution time might be very long, or the solution process might even stop. Stationary problems can run into mesh-resolution issues such as overshooting and undershooting of the solution due to infinite flux problems.

Unless you want to know the details of these transients, start with initial conditions that lead to a consistent solution to a stationary problem. Only then turn on the boundary values, sources, or driving fluxes over a time interval that is realistic for your model.

In most cases you should turn on your sources using a smoothed step over a finite time. What you might think of as a step function is, in real-life physics, often a little bit smoothed because of inertia. The step or switch does not happen instantaneously. Electrical switches take milliseconds, and solid-state switches take microseconds. See "Specifying Discontinuous Functions" on page 149 of the *COMSOL Multiphysics User's Guide* for information about smoothed step functions to try out.

Typographical Conventions

All COMSOL manuals use a set of consistent typographical conventions that should make it easy for you to follow the discussion, realize what you can expect to see on the screen, and know which data you must enter into various data-entry fields. In particular, you should be aware of these conventions:

- A **boldface** font of the shown size and style indicates that the given word(s) appear exactly that way on the COMSOL graphical user interface (for toolbar buttons in the corresponding tooltip). For instance, we often refer to the **Model Navigator**, which is the window that appears when you start a new modeling session in COMSOL; the corresponding window on the screen has the title **Model Navigator**. As another example, the instructions might say to click the **Multiphysics** button, and the boldface font indicates that you can expect to see a button with that exact label on the COMSOL user interface.

- The names of other items on the graphical user interface that do not have direct labels contain a leading uppercase letter. For instance, we often refer to the Draw toolbar; this vertical bar containing many icons appears on the left side of the user interface during geometry modeling. However, nowhere on the screen will you see the term “Draw” referring to this toolbar (if it were on the screen, we would print it in this manual as the **Draw** menu).
- The symbol **>** indicates a menu item or an item in a folder in the **Model Navigator**. For example, **Physics>Equation System>Subdomain Settings** is equivalent to: On the **Physics** menu, point to **Equation System** and then click **Subdomain Settings**. **COMSOL Multiphysics>Heat Transfer>Conduction** means: Open the **COMSOL Multiphysics** folder, open the **Heat Transfer** folder, and select **Conduction**.
- A `Code` (monospace) font indicates keyboard entries in the user interface. You might see an instruction such as “Type `1.25` in the **Current density** edit field.” The monospace font also indicates COMSOL Script codes.
- An *italic* font indicates the introduction of important terminology. Expect to find an explanation in the same paragraph or in the Glossary. The names of books in the COMSOL documentation set also appear using an italic font.

2

Using the Physics Modes

This chapter provides an overview of the physics application modes within COMSOL Multiphysics. It goes on to describe how these application modes make modeling various types of physics easier, faster, and more efficient. Later chapters review each physics mode in detail, including step-by-step examples of how to build and solve models for that particular physics application.

The Physics Modes

One convenient way to solve a physics problem is to set it up with the assistance of templates that COMSOL Multiphysics provides in the form of its *physics modes*. The software computes the PDE coefficients based on application-specific parameters and material properties that you supply, such as Young's modulus for a structural-mechanics model or the heat capacity for a heat-transfer model.

Defining the Physics for a Model

When working with a physics mode, you have access to all necessary settings for material properties, boundary conditions, and other modeling parameters through the **Physics** menu.

SUBDOMAIN SETTINGS AND MATERIAL PROPERTIES

Enter application-specific subdomain properties by choosing **Subdomain Settings** from the **Physics** menu or double-clicking a subdomain in the subdomain selection mode, which you enter by opening the **Subdomain Settings** dialog box, by clicking the **Subdomain Mode** button on the Main toolbar, or on the **Physics** menu by first pointing to **Selection Mode** and then clicking **Subdomain Mode**. For a complete description of the material properties available within each application mode, see the following chapters.

BOUNDARY SETTINGS

To set up boundary conditions, choose **Boundary Settings** from the **Physics** menu or double-click a boundary in the boundary selection mode, which you enter by opening the **Boundary Settings** dialog box, by clicking the **Boundary Mode** button on the Main toolbar, or on the **Physics** menu by first pointing to **Selection Mode** and then click **Boundary Mode**. The **Boundary Settings** dialog box is slightly different for application mode and provides the appropriate boundary conditions. For a complete description of the boundary conditions available for the physics modes, see the following chapters.

POINT SETTINGS

Some models include point sources or use a point in the geometry to fix the value of a variable. To access the point settings, choose **Point Settings** from the **Physics** menu or double-click a point in the point selection mode, which you enter by opening the Point **Settings** dialog box, by clicking the **Point Mode** button on the Main toolbar, or on the

Physics menu by first pointing to **Selection Mode** and then click **Point Mode**. The **Point Settings** dialog box is not available in all application modes.

POSTPROCESSING USING APPLICATION MODE VARIABLES

With the postprocessing mode you can visualize any variable or expression of variables by working with the **Plot Parameters**, **Cross-Section Plot Parameters**, and **Domain Plot Parameters** dialog boxes. You can also have COMSOL Multiphysics compute integrated values with the help of the **Subdomain Integration** and **Boundary Integration** dialog boxes.

Each application mode provides a special set of application-specific variables called *application mode variables*. Lists of these variables for each application mode appear in the following chapters.

Note: For all application modes you have access to shape-function variables and equation variables for the corresponding PDE solution form (coefficient, general, or weak). For further details, please refer to Ref. "Using Variables and Expressions" on page 138 in the *COMSOL Multiphysics User's Guide*.

The Physics Modes

The following table lists the available physics modes:

TABLE 2-1: THE PHYSICS MODES

APPLICATION MODE	DEFAULT Suffix	DEPENDENT VARIABLES	ANALYSIS CAPABILITIES			
			STATIONARY	TIME DEPENDENT	TIME HARMONIC	EIGENFREQUENCY
ACOUSTICS						
Acoustics	aco	p			√	√
HEAT TRANSFER						
Conduction	ht	T	√	√		
Convection and Conduction	cc	T	√	√		
DIFFUSION						

TABLE 2-1: THE PHYSICS MODES

APPLICATION MODE	**DEFAULT** Suffix	**DEPENDENT VARIABLES**	**ANALYSIS CAPABILITIES**			
			STATIONARY	**TIME DEPENDENT**	**TIME HARMONIC**	**EIGENFREQUENCY**
Convection and Diffusion	cd	c	√	√		
Diffusion	di	c	√	√		
ELECTROMAGNETICS						
AC Power Electromagnetics	qa	A_z			√	
Conductive Media DC	dc	V	√			
Electrostatics	es	V	√			
Magnetostatics	qa	A_z	√			
STRUCTURAL MECHANICS						
Solid, Stress-Strain	sld	u, v, w	√	√		√
Plane Stress	ps	u, v	√	√		√
Plane Strain	pn	u, v	√	√		√
Axial Symmetry, Stress-Strain	axi	uor, w	√	√		√
INCOMPRESSIBLE NAVIER STOKES						
Incompressible Navier-Stokes, 2D	ns	u, v, p	√	√		
Incompressible Navier-Stokes, 3D	ns	u, v, w, p	√	√		

Physics Mode Documentation

Chapters 3–8 in this manual give detailed descriptions of each physics mode, typically broken down into the following sections:

VARIABLES AND SPACE DIMENSIONS

The *Variables and Space Dimensions* section describes the dependent and independent variables or space dimensions (1D, 2D, 3D, or axisymmetric) you can use in the application mode.

PDE FORMULATION

The *PDE Formulation* section presents the equations the physics mode solves. It also lists the material properties, sources, and coefficients in the COMSOL Multiphysics formulation.

APPLICATION SCALAR VARIABLES

The *Application Scalar Variables* section lists nongeometric variables specific to the application mode. The default values are physical constants such as the permittivity of vacuum or arbitrary values, for example, the frequency 50 Hz for the AC Power Electromagnetics application mode. Not every application mode has application scalar variables.

SUBDOMAIN SETTINGS

The *Subdomain Settings* section lists the subdomain properties in the application mode such as material properties and sources and sinks.

BOUNDARY CONDITIONS

The *Boundary Conditions* section lists the available boundary conditions while explaining their physical interpretation and the quantities that you specify to define the boundary settings.

LINE AND POINT SETTINGS

In some application modes there are line and point settings for defining line and point sources, pressure constraints, and other properties.

APPLICATION MODE VARIABLES

The *Application Mode Variables* section lists all the variables available to you when formulating the equation and performing postprocessing. You can create functions of these variables for postprocessing and visualizing the analysis and when expressing physical properties. The **Predefined quantities** lists in the dialog boxes for visualization and postprocessing do not always include all the variables; if a variable does not appear in the list, simply type its name in the **Expression** edit field.

Application mode variables are defined on the following geometric domains:

- Subdomains: `S`
- Boundary segments: `B`
- Edges: `E`
- Points: `P`

In the tables that review the application mode variables for each case, the Domain column indicates the top level where the variables are defined. Many variables that are available on subdomains are also available on boundaries, edges, and points, but they then take the average value of the values in the subdomains around the boundary, edge, or point (for the subdomains in which the variable is present). In addition to the letters above, indicating the domains where the application mode variable is valid, a `V` indicates that it is a vector-valued variable.

In some cases the table also lists the solution form for which a variable is available:

- Coefficient form: `c`
- General form: `g`
- Weak form: `w`

The Name column in the table of application mode variables lists the variables available for use, whereas the Expression column lists the implementation of the application mode variables in terms of other variables. The italic *i* and *j* in the variable names can refer to any of the space coordinates. For example, T*i* can mean either `Tx`, `Ty`, or `Tz` in 3D where the space coordinates are x, y, and z. In 2D axisymmetry, T*i* stands for either `Tr` or `Tz`. Further, you construct the variable names of vector components and tensor components using the names of the space coordinates. For example, if an application mode uses `x1`, `y1`, `z1`, then the variables for the vector components of the temperature gradient are `Tx1`, `Ty1`, `Tz1`.

Note: All application mode variables include a suffix indicating which application mode they belong to. Table 2-1 on page 15 lists the default suffix for the physics modes. For example, the default suffix added to application mode variable names for an Incompressible Navier-Stokes application mode is `_ns`.

Note: The default space coordinate names are x, y, z for Cartesian coordinate systems and r, φ (`phi`), and z for cylindrical coordinate systems. It is possible to change these names when defining a new geometry in the Model Navigator. The variable t represents the time for time-dependent models.

3

Acoustics

This chapter describes how to use the Acoustics application mode for modeling and simulation of acoustics and vibrations. It consists of three major sections:

- An overview of acoustics modeling
- A description of the Acoustics application mode
- A step-by-step review of how to model a reactive muffler

Fundamentals of Acoustics

What is Acoustics?

Acoustics is the physics of *sound*. Sound is the sensation, as detected by the ear, of very small rapid changes in the air pressure above and below a static value. This static value is atmospheric pressure (about 100,000 pascals), which varies slowly. Associated with a sound pressure wave is a flow of energy. Physically, sound in air is a longitudinal wave where the wave motion is in the direction of the energy flow. The wave crests are the pressure maxima, while the troughs represent the pressure minima.

Sound results when the air is disturbed by some source. An example is a vibrating object, such as a speaker cone in a hi-fi system. It is possible to see the movement of a bass speaker cone when it generates sound at a very low frequency. As the cone moves forward, it compresses the air in front of it, causing an increase in air pressure. Then it moves back past its resting position and causes a reduction in air pressure. This process continues, radiating a wave of alternating high and low pressure at the speed of sound.

Five Standard Acoustics Problems

Five standard problems or scenarios occur frequently when analyzing acoustics:

- The radiation problem—A vibrating structure (a speaker, for example) radiates sound into the surrounding space. A far-away boundary condition is necessary to model the unbounded domain.
- The scattering problem—An incident wave impinges on a body and creates a scattered wave. A far-away radiation boundary condition is necessary.
- The sound field in an interior space (such as a room)—The acoustic waves stay in a finite volume so no radiation condition is necessary.
- Coupled fluid-elastic structure interaction (structural acoustics)—If the radiating or scattering structure consists of an elastic material, then you must consider the interaction between the body and the surrounding fluid. In the multiphysics coupling, the acoustic analysis provides a load (the sound pressure) to the structural analysis, and the structural analysis provides accelerations to the acoustic analysis.
- The transmission problem—The incident sound wave propagates into a body, which can have different acoustic properties. Pressure and acceleration are continuous on the boundary.

Mathematical Models for Acoustic Analysis

Sound waves in a lossless medium are governed by the following equation for the (differential) pressure p (with SI unit $\mathrm{N/m^2}$):

$$\frac{1}{\rho_0 c_s^2}\frac{\partial^2 p}{\partial t^2} + \nabla \cdot \left(-\frac{1}{\rho_0}(\nabla p - \mathbf{q})\right) = Q$$

Here ρ_0 ($\mathrm{kg/m^3}$) refers to the density and c_s (m/s) denotes the speed of sound. The *dipole source* $\mathbf{q}$ ($\mathrm{N/m^3}$) and the *monopole source* Q ($\mathrm{1/s^2}$) are both optional. The combination $\rho_0 c_s^2$ is called the *bulk modulus*, commonly denoted β ($\mathrm{N/m^2}$).

A special case is a time-harmonic wave, for which the pressure varies with time as

$$p(\mathbf{x}, t) = p(\mathbf{x})e^{i\omega t}$$

where $\omega = 2\pi f$ (rad/s) is the angular frequency, f (Hz) as usual denoting the frequency. Assuming the same harmonic time-dependence for the source terms, the wave equation for acoustic waves reduces to an inhomogeneous Helmholtz equation:

$$\nabla \cdot \left(-\frac{1}{\rho_0}(\nabla p - \mathbf{q})\right) - \frac{\omega^2 p}{\rho_0 c_s^2} = Q \qquad (3\text{-}1)$$

With the source terms removed, you can also treat this equation as an eigenvalue PDE to solve for eigenmodes and eigenfrequencies.

Typical boundary conditions are:

- Sound-hard boundaries (walls)
- Sound-soft boundaries
- Impedance boundary conditions
- Radiation boundary conditions

These are described in more detail below.

In lossy media, an additional term of first order in the time derivative needs to be introduced to model attenuation of the sound waves:

$$\frac{1}{\rho_0 c_s^2}\frac{\partial^2 p}{\partial t^2} - d_a\frac{\partial p}{\partial t} + \nabla \cdot \left(-\frac{1}{\rho_0}(\nabla p - \mathbf{q})\right) = Q$$

The damping term is absent from the standard PDE formulations in the Acoustics application mode. However, in line with COMSOL Multiphysics' general modeling philosophy, the d_a coefficient is accessible from the user interface through the **Subdomain Settings - Equation System** dialog box.

Note also that even when the sound waves propagate in a lossless medium, attenuation frequently occurs by interaction with the surroundings at the boundaries of the system. The tutorial model "Example—Reactive Muffler" on page 34 provides an example of this.

The Acoustics Application Mode

The Acoustics application mode in COMSOL Multiphysics is designed for the analysis of various types of acoustics problems, all concerning pressure waves in a fluid. An acoustics model can be part of a larger multiphysics model that describes, for example, the interactions between structures and acoustic waves.

Variables and Space Dimensions

The Acoustics application mode solves for the acoustic pressure, p. It is available for 2D, 2D axisymmetric, and 3D geometries.

PDE Formulation

COMSOL Multiphysics supplies two PDE formulations—or analysis types—for the Acoustics application mode from which you can choose:

- Time-harmonic analysis
- Eigenfrequency analysis

TIME-HARMONIC ANALYSIS

The time-harmonic—or frequency-domain—formulation is based on the inhomogeneous Helmholtz equation given in Equation 3-1 on page 21 and repeated here for convenience:

$$\nabla \cdot \left(-\frac{1}{\rho_0}(\nabla p - \mathbf{q})\right) - \frac{\omega^2 p}{\rho_0 c_s^2} = Q$$

With this formulation you can compute the frequency response using the parametric solver to sweep over a frequency range using a harmonic load.

EIGENFREQUENCY ANALYSIS

In the eigenfrequency formulation the source terms are absent and you solve for the eigenmodes and the eigenvalues or eigenfrequencies:

$$\nabla \cdot \left(-\frac{1}{\rho_0}\nabla p\right) + \frac{\lambda^2 p}{\rho_0 c_\mathrm{s}^2} = 0 \qquad (3\text{-}2)$$

The eigenvalue λ introduced in this equation is related to the eigenfrequency, f, through $\lambda = -i2\pi f$.

You can switch between specifying the eigenfrequencies and the eigenvalues by choosing **Properties** from the **Physics** menu and changing the value of the property **Specify eigenvalues using** in the **Application Mode Properties** dialog box. There you can also change the analysis type.

Equations 3-1 and 3-2 are both defined in three space dimensions. Because they are given in coordinate-independent notation, the equations apply to 2D, 2D axisymmetric, and 3D models alike. However, upon expanding the ∇ operators, the extra symmetries possessed by 2D and 2D axisymmetric models imply that the equations can be further adapted to the case at hand. For this reason, the general discussion that follows immediately below applies in all its details only to the 3D case, while the particularities of the remaining cases are presented in two separate subsections.

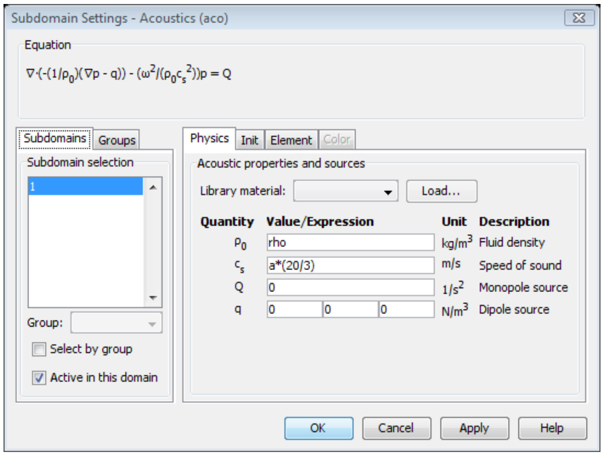
Subdomain Settings

The application-mode characteristic quantities defined on subdomains are:

QUANTITY	VARIABLE	DESCRIPTION
ρ_0	`rho0`	Fluid density
c_s	`cs`	Speed of sound
$\mathbf{q}$	`qx, qy, qz`	Dipole source
Q	`Q`	Monopole source

- In the **Fluid density** edit field you enter the density of the fluid in which the acoustic waves propagate. The default value is 1.25 kg/m^3, the density of air expressed in SI units.
- The **Speed of sound** edit field takes the speed of the sound wave. The default value is 343 m/s, the speed of sound in air at approximately 20 degrees Celsius.
- The **Dipole source** edit fields contain the individual components of the dipole source vector, $\mathbf{q}$, one for each space dimension. The dipole source has the SI unit N/m^3. Its default value is 0.
- The monopole source term Q in the **Monopole source** edit field has the SI unit 1/s^2. It has the default value 0.

You specify these properties in the **Subdomain Settings** dialog box.



Subdomain Settings - Acoustics (aco)

Equation

$\nabla\cdot(-(1/\rho_0)(\nabla p - q)) - (\omega^2/(\rho_0 c_s^2))p = Q$

Subdomains | Groups

Subdomain selection: 1

Group:

☐ Select by group

☑ Active in this domain

Physics | Init | Element | Color

Acoustic properties and sources

Library material: | Load...

Quantity	Value/Expression	Unit	Description
ρ_0	rho	kg/m^3	Fluid density
c_s	a*(20/3)	m/s	Speed of sound
Q	0	1/s^2	Monopole source
q	0 0 0	N/m^3	Dipole source

OK | Cancel | Apply | Help

Boundary Conditions

This section describes the boundary conditions available for the Acoustics application mode and lists the analysis types to which each condition applies.

SOUND-HARD BOUNDARIES (WALLS)

A *sound-hard boundary* is a boundary at which the normal component of the acceleration is zero:

$$-\mathbf{n}\cdot\left(-\frac{1}{\rho_0}(\nabla p - \mathbf{q})\right) = 0$$

For zero dipole charge and constant fluid density, this means that the normal derivative of the pressure is zero at the boundary:

$$\frac{\partial p}{\partial n} = 0$$

Sound-hard boundaries are available for all analysis types.

SOUND-SOFT BOUNDARIES

At a *sound-soft boundary* the differential pressure vanishes:

$$p = 0$$

Sound-soft boundaries are also available for all analysis types.

PRESSURE SOURCE

This boundary condition means that you specify a constant acoustic pressure to be maintained at the boundary:

$$p = p_0$$

The pressure-source condition is available only for time-harmonic analysis.

IMPEDANCE BOUNDARY CONDITION

The impedance boundary condition is a generalization of the sound-hard and sound-soft boundary conditions:

$$\mathbf{n}\cdot\left(-\frac{1}{\rho_0}(\nabla p-\mathbf{q})\right)-\frac{i\omega p}{Z} = 0$$

Here Z (SI unit Pa s/m) is the acoustic input impedance of the external domain. From a physical point of view, the acoustic input impedance is the ratio between pressure and normal particle velocity. It can be expressed in terms of the characteristic impedance inside the domain, $Z_0 = \rho c$, as $Z = \zeta Z_0$, where the dimensionless quantity ζ is called the specific acoustic impedance.

The impedance boundary condition is a good approximation for a locally reacting surface—a surface for which the normal velocity at any point depends only on the pressure at that exact point.

Note that in the two opposite limits $Z \to \infty$ and $Z \to 0$, the sound-hard and sound-soft boundary conditions are recovered. Impedance boundary conditions are available for time-harmonic and transient analysis.

RADIATION BOUNDARY CONDITIONS

The radiation boundary conditions allow an outgoing wave to leave the modeling domain with minimal reflections. In specifying a boundary condition of this kind you have the choice between three wave types: plane, cylindrical, or spherical. You can thus adapt the condition to the geometry of your modeling domain.

The radiation boundary conditions read

$$-\mathbf{n}\cdot\left(-\frac{1}{\rho_0}(\nabla p-\mathbf{q})\right)+(ik+\kappa(r))\frac{p}{\rho_0} = (ik+\kappa(r)-i(\mathbf{k}\cdot\mathbf{n}))\frac{p_0}{\rho_0}e^{-i(\mathbf{k}\cdot\mathbf{r})}$$

where k is the wave number (a predefined application-mode variable; see Table 3-1 on page 30) and $\kappa(r)$ is a function whose form depends on the wave type:

- Plane wave: $\kappa(r) = 0$
- Cylindrical wave: $\kappa(r) = 1/(2r)$
- Spherical wave: $\kappa(r) = 1/r$

In the latter two cases, r is the shortest distance from the point $\mathbf{r} = (x, y, z)$ on the boundary to the source. The right-hand side of the equation represents an optional incoming plane pressure wave with amplitude p_0 and wave vector $\mathbf{k} = k\mathbf{n}_\mathrm{k}$, where $\mathbf{n}_\mathrm{k}$ denotes the unit vector in the direction of propagation.

To define a radiation boundary condition, select a plane, cylindrical, or spherical wave from the **Wave type** list. In addition, you must specify:

- p_0—the pressure source amplitude
- $\mathbf{n}_\mathrm{k}$—the wave direction vector
- $\mathbf{r}_0 = (x_0, y_0, z_0)$—a point on the source axis (for a cylindrical wave) or the source location (for a spherical wave)
- $\mathbf{r}_\mathrm{axis}$—the source axis direction (only for cylindrical waves)

The default value of $\mathbf{n}_\mathrm{k}$ is the inward normal vector, $-\mathbf{n}$, which is the natural direction for waveguides and similar structures. For wave propagation in open space, $\mathbf{k}$ can point in any direction.

Note: You do not have to normalize the vector whose components you enter in the **Wave direction** edit fields; the software explicitly normalizes the components to make $\mathbf{n}_\mathrm{k}$ a unit vector in the direction that you specify.

Radiation boundary conditions are available for time-harmonic analysis only.

SPECIFIED NORMAL ACCELERATION

$$-\mathbf{n} \cdot \left(-\frac{1}{\rho_0}(\nabla p - \mathbf{q})\right) = a_\mathrm{n}$$

The inward normal acceleration a_n represents an external source term. You can also use it for coupling your acoustics model to a structural analysis. The specified normal acceleration is available for time-harmonic analysis.

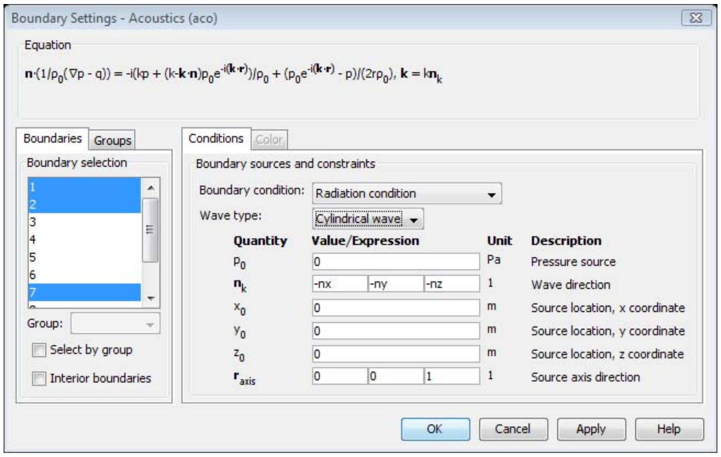
INTERFACE CONDITIONS ON BORDERS

The set of interface conditions for interior boundaries in the Acoustics application mode is:

- *Continuity*

$$\mathbf{n}\cdot\left[\left(-\frac{1}{\rho_0}(\nabla p-\mathbf{q})\right)_1-\left(-\frac{1}{\rho_0}(\nabla p-\mathbf{q})\right)_2\right] = 0$$

- *Sound soft boundaries*—$p = 0$
- *Pressure condition*—$p = p_0$ (not available for eigenvalue analysis)



Boundary Settings - Acoustics (aco)

Equation

$\mathbf{n}\cdot(1/\rho_0(\nabla p - q)) = -i(kp + (k-\mathbf{k}\cdot\mathbf{n})p_0e^{-i(\mathbf{k}\cdot\mathbf{r})})/\rho_0 + (p_0e^{-i(\mathbf{k}\cdot\mathbf{r})} - p)/(2r\rho_0)$, $\mathbf{k} = k\mathbf{n}_k$

Boundaries | Groups

Boundary selection: 1, 2, 3, 4, 5, 6, 7 (1, 2, 7 selected)

Group:

Select by group

Interior boundaries

Conditions | Color

Boundary sources and constraints

Boundary condition: Radiation condition

Wave type: Cylindrical wave

Quantity	Value/Expression	Unit	Description
p_0	0	Pa	Pressure source
$\mathbf{n}_k$	-nx, -ny, -nz	1	Wave direction
x_0	0	m	Source location, x coordinate
y_0	0	m	Source location, y coordinate
z_0	0	m	Source location, z coordinate
$\mathbf{r}_{axis}$	0, 0, 1	1	Source axis direction

Scalar Variables

The following scalar variables appear in the Acoustics application mode, independently of the number of space dimensions:

QUANTITY	VARIABLE	DESCRIPTION	ANALYSIS
f	`freq`	Frequency	H
$i\omega$	`iomega`	Imaginary angular frequency	E
p_{ref}	`p_ref`	Reference pressure	Both

Here H and E stands for time-harmonic analysis and eigenfrequency analysis, respectively.

- *Frequency*—This is an important property for time-harmonic acoustics analysis. It is related to the angular frequency by the equation

$$\omega = 2\pi f$$

The wavelength is given by $\lambda = c_s/f$. Therefore, the higher the frequency, the shorter the wavelength. To resolve a wave, it is important that the mesh size be smaller than the wavelength. As a rule of thumb, use a minimum mesh resolution of a few elements per wavelength with the default second-order Lagrange elements.

Another important acoustics property is the *wave number*, k, which is defined as

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c_s}$$

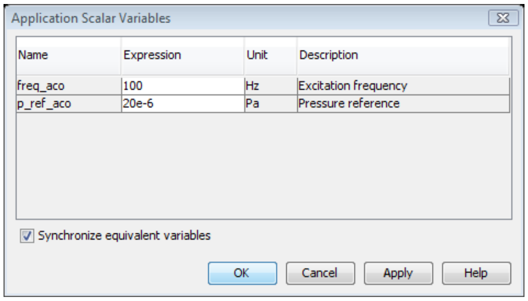
- *Imaginary angular frequency*—For the eigenfrequency analysis type the quantity $i\omega$ is used as a scalar variable in place of the frequency, f. By default, it is related to the eigenvalue, λ, by the equation

  $$i\omega = -\lambda$$

  (Note that the symbol λ is used here in a different meaning than in the previous paragraph.)
- *Reference pressure*—The zero level on the dB scale varies with the type of fluid. The default value (which is the standard value for air) is 0.02 mPa ($20\cdot10^{-6}$ Pa), which then corresponds to 0 dB. This variable occurs only in calculations of the sound pressure level based on the root mean square (rms) pressure,

  $$\bar{p} = \sqrt{\frac{1}{2}p\,\mathrm{conj}(p)}$$

an expression valid for the case of harmonically time-varying acoustic pressure, p.


Application Scalar Variables

Name	Expression	Unit	Description
freq_aco	100	Hz	Excitation frequency
p_ref_aco	20e-6	Pa	Pressure reference

☑ Synchronize equivalent variables

Application Mode Variables

The variables in the following table are available for use in expressions and for postprocessing purposes. Almost all application-mode parameters are available as variables. Some variables are available only in certain analysis types as indicated in the Analysis column, where an H means that the variable is available only for time-harmonic analysis. The table uses an index convention where a single index i denotes any one of the global space variables (x, y, z) while a summation over i runs over all three space dimensions.

TABLE 3-1: ACOUSTICS APPLICATION MODE VARIABLES

NAME	SYMBOL NAME	ANALYSIS	DOMAIN	DESCRIPTION	EXPRESSION
`p`	p	Both	All	pressure	p
`Lp`	L_p	Both	S	sound pressure level	$10\log\left(\frac{\bar{p}^2}{p_0^2}\right)$
`rho0`	ρ_0	Both	S	fluid density	ρ_0
`cs`	c_s	Both	S	speed of sound	c_s
`k`	k	Both	S	wave number	ω/c_s
`q`*`i`*	q_i	Both	S	dipole source, x_i component	q_i
`normq`	$\lvert\mathbf{q}\rvert$	Both	S	dipole source, norm	$\sqrt{\sum_i q_i^2}$
`a`*`i`*	a_i	Both	S	local acceleration, x_i component	$-\frac{1}{\rho_0}\left(\frac{\partial p}{\partial x_i}-q_i\right)$
`norma`	$\lvert\mathbf{a}\rvert$	Both	S	local acceleration, norm	$\sqrt{\sum_i a_i^2}$
`na`	a_n	Both	B	outward normal acceleration	$\mathbf{n}\cdot\left(-\frac{1}{\rho_0}(\nabla p-\mathbf{q})\right)$
`v`*`i`*	v_i	H	S	local velocity, x_i component	$a_i/(i\omega)$
`normv`	$\lvert\mathbf{v}\rvert$	Both	S	local velocity, norm	$\sqrt{\sum_i v_i^2}$
`nv`	v_n	H	B	outward normal velocity	$a_\mathrm{n}/(i\omega)$

TABLE 3-1: ACOUSTICS APPLICATION MODE VARIABLES

NAME	SYMBOL NAME	ANALYSIS	DOMAIN	DESCRIPTION	EXPRESSION
`Ii`	I_i	Both	S	intensity, x_i component	$\mathrm{conj}(v_i)\,p$
`normI`	I	Both	S	intensity, norm	$\sqrt{\sum_i I_i^2}$

To form the complete application mode variable names, add a suffix consisting of an underscore and the application mode name (default: aco), for example, `Lp_aco`. (This does not apply to the dependent variable for the pressure.)

Units

The following table collects the SI units for the most important physical quantities in the Acoustics application mode:

QUANTITY	SYMBOL	SI UNIT	ABBREVIATION
Pressure	p	pascal	Pa
Density	ρ	kilogram/meter^3	kg/m^3
Frequency	f	hertz	Hz
Wave number	k	1/meter	1/m
Dipole source	$\mathbf{q}$	newton/meter^3	N/m^3
Monopole source	Q	1/second^2	1/s^2
Speed of sound	c_s	meter/second	m/s
Acoustic impedance	Z	pascal-second/meter	Pa·s/m
Normal acceleration	a_n	meter/second^2	m/s^2
Source location	r_0	meter	m
Wave direction	$\mathbf{n}_k$	(dimensionless)	1

2D

PDE FORMULATION

In 2D, the independent variables are the Cartesian coordinates x and y, while the model is assumed to be uniform in the perpendicular z direction. Under this assumption, any wave motion in the latter direction is accounted for by specifying the *out-of-plane wave number*, k_z, defined by the equation

$$p(x, y, z, t) = p(x, y)e^{i(\omega t - k_z z)} \tag{3-3}$$

Time-Harmonic Analysis

Using Equation 3-3 and expanding the 3D ∇ operators in Equation 3-1, the equation for the pressure, $p(x, y)$, becomes

$$\nabla \cdot \left[-\frac{1}{\rho_0}(\nabla p - \mathbf{q})\right] - \left[\left(\frac{\omega}{c_s}\right)^2 - k_z{}^2\right]\frac{p}{\rho_0} = Q$$

where the ∇ 's denote 2D differential operators.

Eigenfrequency Analysis

Identical reasoning as for the time-harmonic case leads to the equation

$$\nabla \cdot \left(-\frac{1}{\rho_0}\nabla p\right) + \left[\left(\frac{\lambda}{c_s}\right)^2 + k_z{}^2\right]\frac{p}{\rho_0} = 0$$

SCALAR VARIABLES

In addition to the scalar variables available in the three-dimensional case, in 2D you can also specify the value of the out-of-plane wave number, k_z:

QUANTITY	VARIABLE	DESCRIPTION	ANALYSIS TYPE
f	`freq`	Frequency (time-harmonic)	H
$i\omega$	`iomega`	Imaginary angular frequency	E
k_z	`kz`	Out-of-plane wave number	Both
p_{ref}	`p_ref`	Reference pressure	Both

The default value of k_z is 0.

APPLICATION MODE VARIABLES

With the interpretation that the index i stands for either x or y and that summations run over these two values, all variables listed in Table 3-1 on page 30 are defined also in the 2D case.

2D Axisymmetric

PDE FORMULATION

In the 2D axisymmetric case, the independent variables are the radial coordinate, r, and the axial coordinate, z. The only dependence of the azimuthal coordinate, φ , allowed is through a phase factor:

$$p(r, \varphi, z, t) = p(r, z)e^{i(\omega t - m\varphi)} \tag{3-4}$$

where m denotes the *circumferential wave number*. Because φ is a periodic coordinate, m must be an integer.

Time-Harmonic Analysis

Using Equation 3-4 and expanding the ∇ operators in Equation 3-1, leads to the following equation for the acoustic pressure, $p(r, z)$:

$$\nabla \cdot \left[-\frac{1}{\rho_0}(\nabla p - \mathbf{q})\right] - \left[\left(\frac{\omega}{c_s}\right)^2 - \left(\frac{m}{r}\right)^2\right]\frac{p}{\rho_0} = Q$$

Eigenfrequency Analysis

The corresponding eigenvalue equation reads

$$\nabla \cdot \left(-\frac{1}{\rho_0}\nabla p\right) + \left[\left(\frac{\lambda}{c_s}\right)^2 + \left(\frac{m}{r}\right)^2\right]\frac{p}{\rho_0} = 0$$

BOUNDARY CONDITIONS

Axial Symmetry

This condition is available only for axisymmetric models, where it applies to the symmetry boundary $r = 0$.

SCALAR VARIABLES

Beside the scalar variables common with the 3D and 2D cases, you can specify the value of the circumferential wave number, m, in the 2D axisymmetric case:

QUANTITY	VARIABLE	DESCRIPTION	ANALYSIS TYPE
f	`freq`	Frequency	H
$i\omega$	`iomega`	Imaginary angular frequency	E
m	`m`	Circumferential wave number	Both
p_{ref}	`p_ref`	Reference pressure for dB computation	Both

The default value of m is 0.

APPLICATION MODE VARIABLES

With the interpretation that the index i stands for either r or z and that summations run over these two values, Table 3-1 on page 30 applies also to the 2D axisymmetric case.

Example—Reactive Muffler

Introduction

This model examines the sound-transmission properties of an idealized reactive muffler with infinitely long inlet and outlet pipes (or a reflection-free source at the inlet pipe and a reflection-free end of the outlet pipe) and one expansion chamber. One measure of the transmission properties is the transmission-loss coefficient, D_{tl}, which is defined as

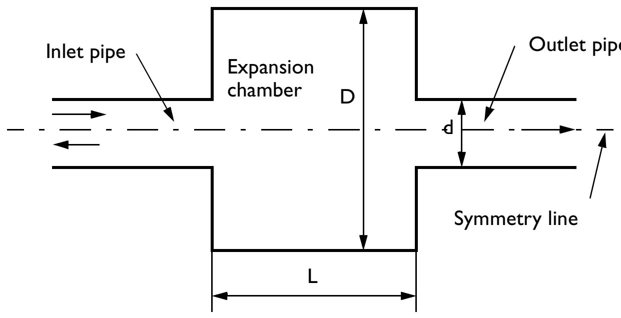
$$D_{\mathrm{tl}} = 10 \cdot \log\left(\frac{W_{\mathrm{i}}}{W_{\mathrm{t}}}\right)$$

where W_{i} is the time-averaged incident sound power and W_{t} is the transmitted sound power. This problem has a theoretical 1D solution that you can compare with the FEM solution.

Model Definition

In the following figure, a plane sound wave enters the inlet pipe (left) and is reflected and attenuated in the expansion chamber. The attenuated sound wave exits through the outlet pipe (right).

The diameter of both the inlet pipe and the outlet pipe is d, and the corresponding cross-sectional area is S_1. The expansion chamber has a diameter D with a corresponding cross-sectional area S_2.



According to Ref. 1, the 1D theoretical solution for the transmission loss to this problem is

$$D_{\mathrm{tl}} = 10 \cdot \log\left[1 + \left(\frac{S_1}{2 \cdot S_2} - \frac{S_2}{2 \cdot S_1}\right)^2 \cdot (\sin(kL)^2)\right]$$

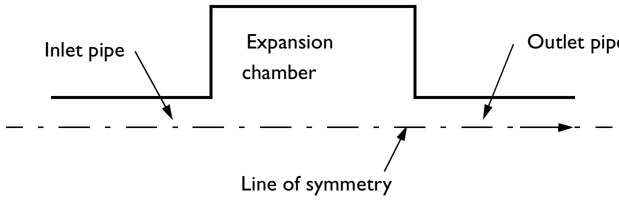
where k is the wave number; S_1 and S_2 are the areas of the pipes and expansion chamber; and L gives the length of the expansion chamber.

The model computes the pressure, p, for the fluid in the region defined by the above geometry. This is a time-harmonic problem so you can use the Helmholtz equation defined in the axisymmetric Acoustics application mode:

$$\nabla \cdot \left(-\frac{1}{\rho_0}(\nabla p - \mathbf{q})\right) - \frac{\omega^2 p}{\rho_0 c_s^2} = 0$$

where $\omega = 2\pi f$ is the angular frequency, ρ_0 is the fluid density, and c_s is the speed of sound. The $\mathbf{q}$ term is a dipole source with the dimension of force per volume.

Because this is an axisymmetric model, you need to include only half of the geometry as indicated in the following figure:



You must apply axial symmetry boundary conditions on the line of symmetry.

Assume the walls are rigid, and thus use sound-hard (wall) boundary conditions,

$$\frac{\partial p}{\partial n} = 0$$

which means that the normal derivative of the pressure is zero at the boundaries.

Radiation boundary conditions describe the inlet and outlet boundaries:

$$-\mathbf{n} \cdot \left(-\frac{1}{\rho_0}(\nabla p - \mathbf{q})\right) + \left(\frac{ik}{\rho_0}\right)p = \frac{(ik - i(\mathbf{k} \cdot \mathbf{n}))p_0 e^{-i(\mathbf{k} \cdot \mathbf{r})}}{\rho_0}$$

The radiation boundary condition is useful when the surroundings are merely a continuation of the domain, which is the case in this model. The term on the right-hand side represents an incoming pressure wave with an amplitude p_0 and a direction given by the wave vector, **k**. In this model, an incoming pressure wave with the amplitude p_0 = 1 Pa enters at the inlet boundary.

To determine the transmission loss in the model, you must first calculate the incident and transmitted time-averaged sound intensities and the corresponding sound power values. The equation

$$I = \frac{p^2}{2\rho_0 c}$$

gives the time-averaged sound intensities where p is equal to p_0 at the inlet and the computed solution at the outlet.

Using the boundary integration tool, you can evaluate the incident and transmitted sound powers, W, as:

$$W = \int (I \cdot 2\pi r) dr$$

Results and Discussion

Figure 3-1 shows the theoretical transmission loss (square markers) and the COMSOL Multiphysics solution (triangle markers) as a function of frequency. The theoretical solution has an upper frequency limit for its validity. This limit is the cut-on frequency, which defines the frequency range where only plane waves can propagate; above this frequency, also higher modes can propagate.

According to Ref. 1, the first cut-on frequency for a pipe is

$$f_{01} = 1.841 \frac{c}{\pi D}.$$

Its value in this case is approximately 332 Hz, but it is evident from the above figure that a discrepancy exists between the theoretical and the FEM solution, even below the cut-on frequency. The discrepancy also increases with frequency between the 1D theoretical model and a 3D analysis, as you can see in Ref. 1. In the lower frequency range, however, there is good agreement between the theoretical solution and the FEM solution.

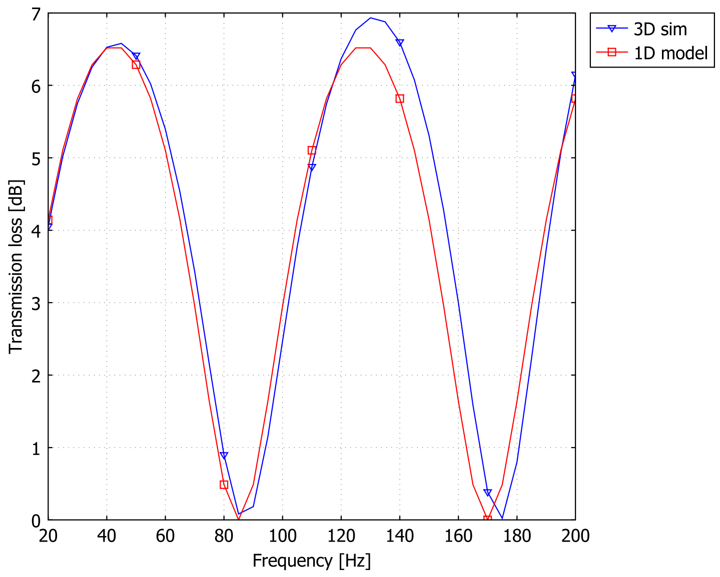


Figure 3-1: Muffler transmission loss versus frequency: theoretical solution (squares) and COMSOL Multiphysics solution (triangles).

Reference

1. H.P. Wallin, *Ljud och Vibrationer,* Institutionen för Farkostteknik, KTH, Stockholm, Sweden, 1999 (in Swedish).

Model Library path: `COMSOL_Multiphysics/Acoustics/reactive_muffler`

Modeling Using the Graphical User Interface

MODEL NAVIGATOR

1 Go to the **Model Navigator** and select **Axial symmetry (2D)** in the **Space dimension** list.

2 In the list of application modes open the **COMSOL Multiphysics>Acoustics>Acoustics** folder and then select **Time-harmonic analysis**.

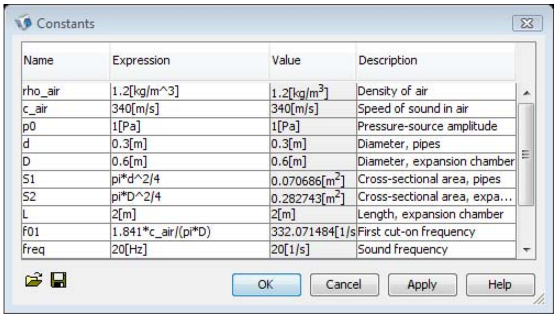
3 Click **OK**.

OPTIONS AND SETTINGS

1 Go to the **Options** menu and choose **Constants** to parameterize the model.

2 In the **Constants** dialog box enter the following constants, representing fluid properties and some geometrical properties to calculate the cut-on frequency and the theoretical transmission loss:

NAME	EXPRESSION	DESCRIPTION
`rho_air`	`1.2[kg/m^3]`	`Density of air`
`c_air`	`340[m/s]`	`Speed of sound in air`
`p0`	`1[Pa]`	`Pressure-source amplitude`
`d`	`0.3[m]`	`Diameter, pipes`
`D`	`0.6[m]`	`Diameter, expansion chamber`
`S1`	`pi*d^2/4`	`Cross-sectional area, pipes`
`S2`	`pi*D^2/4`	`Cross-sectional area, expansion chamber`
`L`	`2[m]`	`Length, expansion chamber`
`f01`	`1.841*c_air/(pi*D)`	`First cut-on frequency`
`freq`	`20[Hz]`	`Sound frequency`


Constants

Name	Expression	Value	Description
rho_air	1.2[kg/m^3]	1.2[kg/m^3]	Density of air
c_air	340[m/s]	340[m/s]	Speed of sound in air
p0	1[Pa]	1[Pa]	Pressure-source amplitude
d	0.3[m]	0.3[m]	Diameter, pipes
D	0.6[m]	0.6[m]	Diameter, expansion chamber
S1	pi*d^2/4	0.070686[m^2]	Cross-sectional area, pipes
S2	pi*D^2/4	0.282743[m^2]	Cross-sectional area, expa...
L	2[m]	2[m]	Length, expansion chamber
f01	1.841*c_air/(pi*D)	332.071484[1/s	First cut-on frequency
freq	20[Hz]	20[1/s]	Sound frequency

GEOMETRY MODELING

1 Shift-click the **Rectangle/Square** button to specify a rectangle.

2 Go to the **Rectangle** dialog box and type `0.3` in the **Width** edit field and `1` in the **Height** edit field.

3 Click **OK**.

4 Click the **Zoom Extents** button.

5 Shift-click the **Rectangle/Square** button to specify another rectangle.

6 In the **Rectangle** dialog box, type `0.6` in the **Width** edit field, `2` in the **Height** edit field, and `1` in the **z** edit field. Click **OK**.

7 Shift-click the **Rectangle/Square** button to specify a third rectangle.

8 In the **Rectangle** dialog box type `0.3` in the **Width** edit field, `1` in the **Height** edit field, and `3` in the **z** edit field. Click **OK**.

9 Click the **Zoom Extents** button on the Main toolbar.

PHYSICS SETTINGS

Subdomain Settings

This model uses the fluid properties of air, specified in SI units.

Enter these quantities:

QUANTITY	ALL SUBDOMAINS
ρ_0	`rho_air`
c_s	`c_air`

1 From the **Physics** menu choose **Subdomain Settings**.

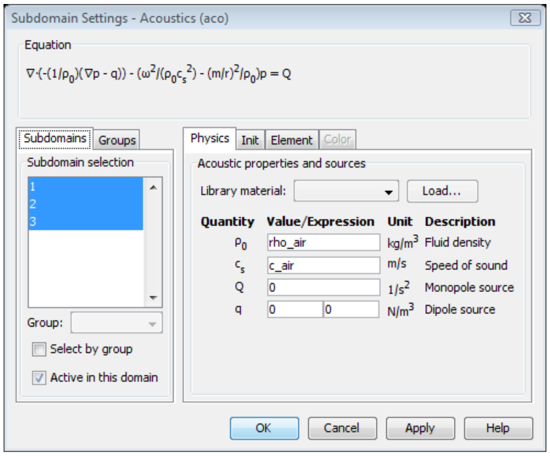
2 In the **Subdomain Settings** dialog box select all subdomains from the **Subdomain selection** list.

3 Type `rho_air` in the **Fluid Density** edit field.

4 Type `c_air` in the **Speed of sound** edit field.

5 Leave the default settings (all 0) for the **Dipole source** and the **Monopole source**.

6 Click **OK**.



Subdomain Settings - Acoustics (aco)

Equation

$\nabla\cdot(-(1/\rho_0)(\nabla p - q)) - (\omega^2/(\rho_0 c_s^2) - (m/r)^2/\rho_0)p = Q$

Subdomain selection: 1, 2, 3

Acoustic properties and sources

Library material:

Quantity	Value/Expression	Unit	Description
ρ_0	rho_air	kg/m^3	Fluid density
c_s	c_air	m/s	Speed of sound
Q	0	1/s^2	Monopole source
q	0 0	N/m^3	Dipole source

Select by group [ ]

Active in this domain [✓]

Boundary Conditions

1 From the **Physics** menu choose **Boundary Settings**.

2 In the **Boundary Settings** dialog box enter the following boundary condition types and properties:

SETTINGS	BOUNDARIES 1, 3, 5	BOUNDARIES 8–12	BOUNDARY 2	BOUNDARY 7
Boundary condition	Axial symmetry	Sound hard boundary (wall)	Radiation condition	Radiation condition
Wave type			Plane wave	Plane wave
p_0			1	0

3 Select Boundaries 1, 3, and 5 in the **Boundary selection** list.

4 Select **Axial symmetry** in the **Boundary condition** list.

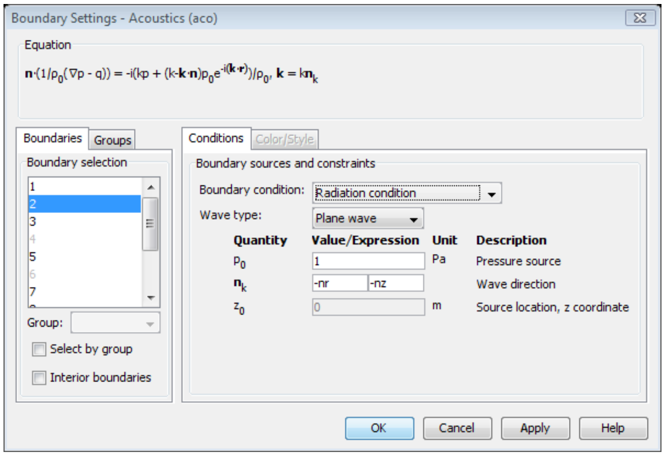
5 Select Boundaries 8–12 in the **Boundary selection** list.

6 Select **Sound hard boundary (wall)** in the **Boundary condition** list.

7 Select Boundary 2 in the **Boundary selection** list.

8 Select **Radiation condition** in the **Boundary condition** list.

9 Type 1 in the **Pressure source** edit field.


Boundary Settings - Acoustics (aco)

Equation

$\mathbf{n}\cdot(1/\rho_0(\nabla p - q)) = -i(kp + (k-\mathbf{k}\cdot\mathbf{n})p_0 e^{-i(\mathbf{k}\cdot\mathbf{r})})/\rho_0,\ \mathbf{k} = k\mathbf{n}_k$

Boundaries | Groups

Boundary selection: 1, 2 (selected), 3, 4, 5, 6, 7

Group:

Select by group

Interior boundaries

Conditions | Color/Style

Boundary sources and constraints

Boundary condition: Radiation condition

Wave type: Plane wave

Quantity	Value/Expression	Unit	Description
p_0	1	Pa	Pressure source
$\mathbf{n}_k$	-nr, -nz		Wave direction
z_0	0	m	Source location, z coordinate

OK | Cancel | Apply | Help

10 Finally select Boundary 7 in the **Boundary selection** list.

11 Select **Radiation condition** in the **Boundary condition** list.

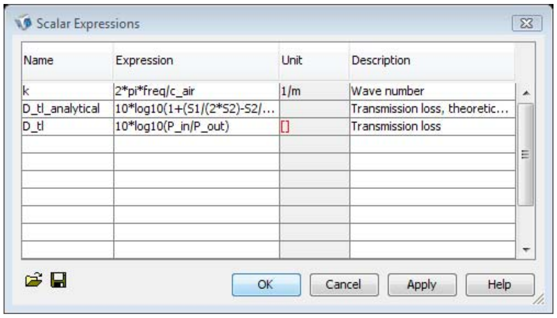
12 Click **OK**.

Expression Variables

1 On the **Options** menu, point to **Expressions**, and then click **Scalar Expressions**.

2 In the **Scalar Expressions** dialog box enter the following:

NAME	EXPRESSION	DESCRIPTION
`k`	`2*pi*freq/c_air`	`Wave number`
`D_tl_analytical`	`10*log10(1+(S1/(2*S2)-S2/(2*S1))^2*(sin(k*L)).^2)`	`Transmission loss, theoretical 1D model`
`D_tl`	`10*log10(P_in/P_out)`	`Transmission loss`



Scalar Expressions

Name	Expression	Unit	Description
k	2*pi*freq/c_air	1/m	Wave number
D_tl_analytical	10*log10(1+(S1/(2*S2)-S2/...		Transmission loss, theoretic...
D_tl	10*log10(P_in/P_out)	[]	Transmission loss

The red brackets in the **Unit** column for `D_tl` appear because `P_in` and `P_out`, which you define as integration coupling variables shortly, do not have units. Because the expression is, nevertheless, dimensionally correct, you can ignore this warning.

3 Click **OK**.

4 Go to the **Options** menu and choose **Expressions** and then **Boundary Expressions**.

5 In the **Boundary Expressions** dialog box select Boundary 2 from the **Boundary selection** list and enter the following boundary expression variable:

NAME	EXPRESSION
`I_in`	`real(conj(p0)*p0)/(2*rho_air*c_air)`

6 In the **Boundary Expressions** dialog box select Boundary 7 from the **Boundary selection** list and enter the following boundary expression variable:

NAME	EXPRESSION
`I_in`	
`I_n`	`real(conj(p)*p)/(2*rho_air*c_air)`

7 Click **OK**.

Integration Coupling Variables

1 Go to the **Options** menu and choose **Integration Coupling Variables** and then **Boundary Variables**.

2 In the **Boundary Integration Variables** dialog box select Boundary 2 and then enter the following boundary integration expression:

NAME	EXPRESSION	INTEGRATION ORDER
`P_in`	`I_in*2*pi*r`	4

3 In the **Boundary Integration Variables** dialog box select Boundary 7 and enter the following boundary integration expression; when done, click **OK**.

NAME	EXPRESSION	INTEGRATION ORDER
`P_in`		
`P_out`	`I_n*2*pi*r`	4

MESH GENERATION

1 Go to the **Mesh** menu and choose **Free Mesh Parameters**.

2 In the **Free Mesh Parameters** dialog box, select **Finer** in the **Predefined mesh sizes** list.

3 Click **Remesh**, then click **OK**.

COMPUTING THE SOLUTION

1 From the **Solve** menu, choose **Solver Parameters**.

2 In the **Solver Parameters** dialog box, select **Parametric** from the **Solver** list.

3 Type `freq` in the **Parameter name** edit field.

4 Type `20:5:200` in the **Parameter values** edit field.

5 Click **OK**.

6 Go to the **Physics** menu and choose **Scalar Variables**.

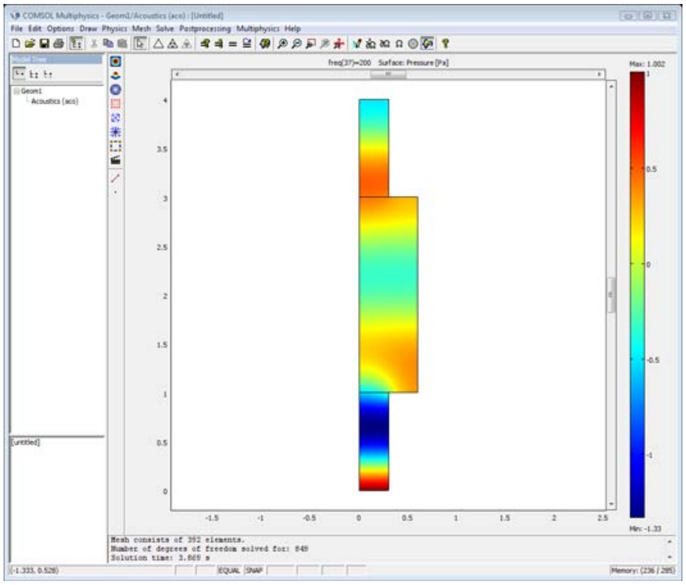
7 In the **Expression** column, type `freq` in the edit field for the frequency.

8 Click **OK**.

9 Click the **Solve** button to start the simulation.

POSTPROCESSING AND VISUALIZATION

The default visualization plots the magnitude of the pressure field at the final frequency (200 Hz).


COMSOL Multiphysics - Geom1/Acoustics (aco) : [Untitled]
File Edit Options Draw Physics Mesh Solve Postprocessing Multiphysics Help
freq(37)=200 Surface: Pressure [Pa]
Max: 1.002
Min: -1.33
Geom1
Acoustics (aco)
EQUAL
SNAP

```
Mesh consists of 352 elements.
Number of degrees of freedom solved for: 849
Solution time: 3.869 s
```

Next generate the transmission-loss plots in Figure 3-1.

1 Go to the **Postprocessing** menu and select **Domain Plot Parameters**.
2 On the **General** page, select all frequencies from the **Solutions to use** list in the **Domain Plot Parameters** dialog box.
3 Select the **Keep current plot** check box.
4 On the **Point** page, select Point 1 from the **Point selection** list.
5 Type `D_tl` in the **Expression** edit field.
6 Click the **Line Settings** button and select **Triangle** in the **Line marker** list. Click **OK**.
7 Click **Apply** in the **Domain Plot Parameters** dialog box.

   To make it easy to compare the two solutions, plot the theoretical solution in the same figure.

8 Type `D_tl_analytical` in the **Expression** edit field.
9 Click the **Line Settings** button. Select **Color** from the **Line color** list and **Square** from the **Line marker** list. Click **OK**.
10 Click **OK** in the **Domain Plot Parameters** dialog box
11 In the figure window, click the **Edit Plot** toolbar button. Finish the plot by editing the plot title and axis labels, and adding labels.

4

Diffusion

This chapter explains how to use both the Diffusion application mode and the Diffusion and Convection application mode to model and simulate various types of transport problems. It concludes with a step-by-step example that introduces the concept of effective diffusivity in porous media.

The Diffusion Application Mode

The Diffusion application mode treats a specific mechanism of mass transport, the simplest one to describe in mathematical terms, namely diffusion. This application mode is available in 1D, 2D, and 3D as well as in axisymmetric formulations in 1D and 2D.

The dependent variable is the concentration of mass, c.

Note: The optional Chemical Engineering Module provides an application mode for the diffusion of several species as well as an application mode for problems that involve potential flow. It also supports modeling using pseudo-2D and pseudo-3D geometries.

PDE Formulation

Heat transfer is a diffusion process, so the generic diffusion equation has the same structure as the heat equation. Diffusion is governed by the equation

$$\delta_{\mathrm{ts}}\frac{\partial c}{\partial t} + \nabla \cdot (-D\nabla c) = R$$

where c is the concentration, D is the *diffusion coefficient*, and R is a reaction rate. The diffusion process can be anisotropic, in which case D is a tensor.

Subdomain Settings

The diffusion equation coefficients to specify on the subdomains are:

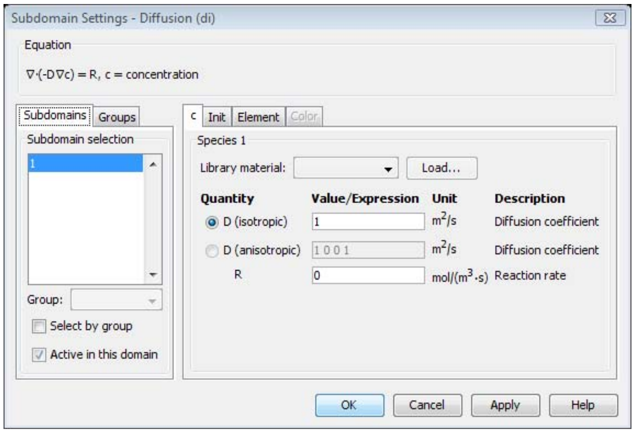
COEFFICIENT	VARIABLE	DESCRIPTION
δ_{ts}	`Dts_c`	Time-scaling coefficient
D	`D_c`	Diffusion coefficient, diffusion coefficient tensor
D_{ij}	D$xixj$_c	Diffusion coefficient tensor, x_ix_j component
R	`R_c`	Reaction rate

Time-scaling Coefficient This coefficient is normally 1. If desired, you can change the time scale, for example, from seconds to minutes by setting it to 1/60.

Diffusion Coefficient This material property, denoted by D, describes a material's diffusive conductivity, in other words the relation between the concentration gradient and the mass flux.

Reaction Rate The reaction rate describes the volume density of mass creation. The reaction rate can be nonlinear.

You specify the equation coefficients in the **Subdomain Settings** dialog box.



Subdomain Settings - Diffusion (di)

Equation

∇·(-D∇c) = R, c = concentration

Subdomains | Groups

Subdomain selection: 1

Group:

Select by group

Active in this domain: ✓

c | Init | Element | Color

Species 1

Library material: Load...

Quantity	Value/Expression	Unit	Description
● D (isotropic)	1	m^2/s	Diffusion coefficient
○ D (anisotropic)	1 0 0 1	m^2/s	Diffusion coefficient
R	0	$mol/(m^3 \cdot s)$	Reaction rate

OK | Cancel | Apply | Help

For equations in 2D or 3D, pay special attention to the isotropic diffusion coefficient, D. If you select this coefficient, the application mode expands it to the diagonal of the diffusion coefficient tensor, that is, Dx_ix_i equals D.

Boundary Conditions

The boundary conditions are:

BOUNDARY CONDITION	DESCRIPTION
$c = c_0$	Concentration
$-\mathbf{n} \cdot (-D\nabla c) = N_0 + k_c(c_b - c)$	Flux
$\mathbf{n} \cdot (-D\nabla c) = 0$	Insulation/Symmetry

BOUNDARY CONDITION	DESCRIPTION
$\mathbf{n}_1 \cdot (-D\nabla c)_1 = \frac{D}{d}(c_1 - c_2)$ $\mathbf{n}_2 \cdot (-D\nabla c)_2 = \frac{D}{d}(c_2 - c_1)$	Thin boundary layer
$\mathbf{n} \cdot (\mathbf{N}_1 - \mathbf{N}_2) = 0$	Continuity
$-\mathbf{n} \cdot (\mathbf{N}_1 - \mathbf{N}_2) = N_0$	Flux discontinuity
$\mathbf{n} \cdot (-D\nabla c) = 0$	Axial symmetry

Concentration In the equation for the concentration boundary condition, c_0 is a user-specified concentration.

Diffusive Flux In the equation for the flux condition, `kc` represents the mass transfer coefficient, `cb` is the bulk concentration, and N_0 is an arbitrary user-specified flux expression.

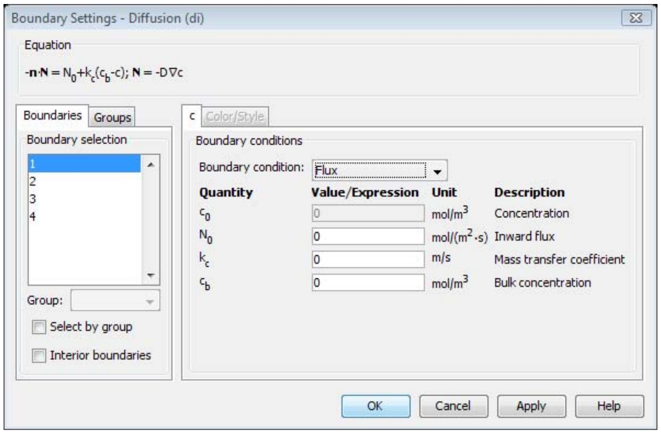
Thin Boundary Layer You can use the thin boundary layer condition to model a thin layer of a material with a small diffusion coefficient compared to the adjacent domains. The layer has the thickness d and the diffusion coefficient D. This boundary condition is only available at the border between the parts in an assembly.

Flux Discontinuity This boundary condition represents a discontinuity in the mass flux across an interior boundary or a border between parts in an assembly; it is not applicable to exterior boundaries.

Continuity This is the default boundary condition on interior boundaries and pair boundaries; it is not applicable to exterior boundaries.

Axial Symmetry The axial symmetry condition is identical to the insulation/symmetry condition. It is available only for axisymmetric models using cylindrical coordinate systems. Use this boundary condition on the symmetry axis.

Specify the boundary conditions in the **Boundary Settings** dialog box. For boundary conditions on pair boundaries, click the **Pair** tab above the selection area in the left part of the dialog box.



Boundary Settings - Diffusion (di)

Equation

$-\mathbf{n}\cdot\mathbf{N} = N_0 + k_c(c_b - c)$; $\mathbf{N} = -D\nabla c$

Boundaries | Groups

Boundary selection: 1, 2, 3, 4

Group:

Select by group

Interior boundaries

c | Color/Style

Boundary conditions

Boundary condition: Flux

Quantity	Value/Expression	Unit	Description
c_0	0	mol/m^3	Concentration
N_0	0	mol/(m^2·s)	Inward flux
k_c	0	m/s	Mass transfer coefficient
c_b	0	mol/m^3	Bulk concentration

OK | Cancel | Apply | Help

Application Mode Variables

The Diffusion application mode uses the following expressions and coefficients in boundary conditions, equations, and for postprocessing purposes.

NAME	TYPE	DESCRIPTION	EXPRESSION
`c`	S/B	Concentration	c
`grad_c`, `c`*`xi`*	S/V	Concentration gradient	$\lvert\nabla c\rvert,\ \frac{\partial c}{\partial x_i}$
`dflux_c`	S	Diffusive flux	$\lvert\underline{D}\nabla c\rvert$
`ndflux_c`	B	Normal diffusive flux	$\mathbf{n}\cdot(-\underline{D}\nabla c)$
`dflux_c_`*`xi`*	V	Diffusive flux, x_i component	$\sum_j -D_{ij}\frac{\partial c}{\partial x_j}$
`Dts_c`	S	Time-scaling coefficient	δ_{ts}
`D_c`, `D`*`xixj`*`_c`	S	Diffusion coefficient	D, D_{ij}
`R_c`	S	Reaction rate	R
`N_c`	B	Inward diffusive flux	N_0
`kc_c`	B	Mass-transfer coefficient	k_c

NAME	TYPE	DESCRIPTION	EXPRESSION
cb_c	B	Bulk concentration	c_b
c0_c	B	Concentration	c_0

The vector expressions indicated with a V in the Type column are not present in 1D versions of the diffusion application mode.

Note: To form the complete application mode variable names, add a suffix consisting of an underscore and the application mode name (default: di), for example, `dflux_d_di`. (This does not apply to the dependent variable for the concentration.)

The Convection and Diffusion Application Mode

This application mode models the most common type of transport in chemical systems: transport by convection and diffusion. You can simulate transport by convection and diffusion in 1D, 2D, and 3D as well as for axisymmetric systems in 1D and 2D.

The dependent variable in the application mode is the concentration of mass, c.

Note: The optional Chemical Engineering Module contains an application mode for the convection and diffusion of several species. It also supports modeling using pseudo-2D and pseudo-3D geometries.

PDE Formulation

The equations for the nonconservative and conservative formulations for a species, c, are:

$$\delta_{\mathrm{ts}}\frac{\partial c}{\partial t} + \nabla \cdot (-D\nabla c) = R - \mathbf{u} \cdot \nabla c \quad \text{nonconservative}$$

$$\delta_{\mathrm{ts}}\frac{\partial c}{\partial t} + \nabla \cdot (-D\nabla c + c\mathbf{u}) = R \quad \text{conservative}$$

The nonconservative formulation is the default for advection and diffusion types of equations in COMSOL Multiphysics because the software assumes an incompressible fluid. Thus the term $c\nabla \cdot \mathbf{u}$ equals zero and gets dropped from the nonconservative formulation. This ensures that no nonphysical source term arises from a flow field where the incompressibility constraint, $\nabla \cdot \mathbf{u} = 0$, is not absolutely fulfilled.

For stationary analysis the term with the time derivative gets dropped.

Subdomain Settings

The various coefficients used in the equations are:

COEFFICIENT	VARIABLE	DESCRIPTION
δ_{ts}	`Dts_c`	Time-scaling coefficient
D	`D_c`	Diffusion coefficient or tensor
D_{ij}	D*xixj*_c	Diffusion coefficient tensor, $x_i x_j$ component
R	`R_c`	Reaction rate
u, v, w	`u_c`, `v_c`, `w_c`	Velocity in the x_1, x_2, and x_3 directions

For equations in 2D or 3D, pay special attention to the isotropic diffusion coefficient, D. If you select this coefficient, the application mode expands it to the diagonal of the diffusion coefficient tensor, that is, Dx_ix_i equals D.

You specify the equation coefficients in the **Subdomain Settings** dialog box.

ARTIFICIAL DIFFUSION

The Convection and Diffusion application mode supports artificial diffusion using the following methods:

- Isotropic diffusion
- Streamline diffusion
- Crosswind diffusion

To specify and activate artificial diffusion:

1. Open the **Subdomain Settings** dialog box.
2. Click the **Physics** tab.
3. With at least one subdomain selected, click the **Artificial Diffusion** button.

See “Stabilization Techniques” on page 433 in the *COMSOL Multiphysics Modeling Guide* for more information about artificial diffusion.

Boundary Conditions

The available boundary conditions are:

BOUNDARY CONDITION	DESCRIPTION
$c = c_0$	Concentration
$-\mathbf{n} \cdot (-D\nabla c + c\mathbf{u}) = N_0$	Flux
$\mathbf{n} \cdot (-D\nabla c + c\mathbf{u}) = 0$	Insulation/Symmetry
$\mathbf{n} \cdot (-D\nabla c) = 0$	Convective flow
$\mathbf{n}_1 \cdot (-D\nabla c + c\mathbf{u})_1 = \frac{D}{d}(c_1 - c_2)$ $\mathbf{n}_2 \cdot (-D\nabla c + c\mathbf{u})_2 = \frac{D}{d}(c_2 - c_1)$	Thin boundary layer
$\mathbf{n} \cdot (\mathbf{N}_1 - \mathbf{N}_2) = 0$	Continuity
$-\mathbf{n} \cdot (\mathbf{N}_1 - \mathbf{N}_2) = N_0$	Flux discontinuity
$\mathbf{n} \cdot (-D\nabla c + c\mathbf{u}) = 0$	Axial symmetry

Concentration In the equation for the concentration boundary condition, c_0 is a user-specified concentration.

Diffusive Flux In the equation for the flux condition, N_0 is an arbitrary user-specified flux expression.

Thin Boundary Layer You can use the thin boundary layer condition to model a thin layer of a material with a small diffusion coefficient compared to the adjacent domains. The layer has the thickness d and the diffusion coefficient D. This boundary condition is only available at the border between the parts in an assembly.

Flux Discontinuity This boundary condition represents a discontinuity in the mass flux across a border between parts in an assembly.

Continuity This is the default boundary condition on interior boundaries and pair boundaries; it is not applicable to exterior boundaries.

Axial Symmetry The axial symmetry condition is identical to the insulation/symmetry condition. It is available only for axisymmetric models using cylindrical coordinate systems. Use this boundary condition only on the symmetry axis.

You specify boundary conditions in the **Boundary Settings** dialog box. For boundary conditions on pair boundaries, click the **Pair** tab above the selection area in the left part of the dialog box.

Application Mode Variables

The Convection and Diffusion application mode uses the following expressions and coefficients in boundary conditions, equations, and for postprocessing purposes:

NAME	TYPE	DESCRIPTION	EXPRESSION
`c`	S/B	Concentration	c
`grad_c`, `cx`*i*	S/V	Concentration gradient	$\lvert\nabla c\rvert,\quad \frac{\partial c}{\partial x_i}$
`dflux_c`	S	Diffusive flux	$\lvert D\nabla c\rvert$
`cflux_c`	S	Convective flux	$\lvert c\mathbf{u}\rvert$
`tflux_c`	S	Total flux	$\lvert -D\nabla c + c\mathbf{u}\rvert$
`ndflux_c`	B	Normal diffusive flux	$\mathbf{n}\cdot(-D\nabla c)$
`ncflux_c`	B	Normal convective flux	$c\mathbf{n}\cdot\mathbf{u}$
`ntflux_c`	B	Normal total flux	$\mathbf{n}\cdot(-D\nabla c + c\mathbf{u})$
`dflux_c_x`*i*	V	Diffusive flux, x_i component	$\sum_j -D_{ij}\frac{\partial c}{\partial x_j}$
`cflux_c_x`*i*	V	Convective flux, x_i component	cu_i
`tflux_c_x`*i*	V	Total flux, x_i component	$\sum_j -D_{ij}\frac{\partial c}{\partial x_j} + cu_i$
`cellPe_c`	S	Cell Peclet number	$\left\lvert\frac{\mathbf{u}h}{D}\right\rvert$
`Dts_c`	S	Time-scaling coefficient	δ_{ts}
`udl_c`	S	Dimensionless velocity	u_{dl}
`D_c`, `Dx`*i*`x`*j*`_c`	S	Diffusion coefficient	D, D_{ij}
`R_c`	S	Reaction rate	R
`u_c`, `v_c`, `w_c`	S	Velocity of c, x_i component	u_i
`N_c`	B	Inward flux	N_0
`c0_c`	B	Concentration	c_0

NAME	TYPE	DESCRIPTION	EXPRESSION
beta_c_x*i*	S	Convective field, x_i component	u_i
Dm_c	S	Mean diffusion coefficient	$\frac{\sum_{i,j} D_{ij}\beta_i\beta_j}{\lvert\beta\rvert}$
res_c	S	Equation residual	$\nabla\cdot(-D\nabla c+c\mathbf{u})-R$
res_sc_c	S	Shock capturing residual	$\nabla\cdot(c\mathbf{u})-R$
da_c	S	Total time-scale factor	δ_{ts}

The vector expressions indicated with a V in the Type column are not present in 1D versions of the Convection and Diffusion application mode.

Note: To form the complete application mode variable names, add a suffix consisting of an underscore and the application mode name (default: cd), for example, `dflux_d_cd`. (This does not apply to the dependent variable for the concentration.)

Effective Diffusivity in Porous Materials

This model introduces the concept of effective diffusivity in porous media by comparing the transport through an artificial porous structure described in a detailed model with a simplified homogeneous porous media approach using effective transport properties.

The exercise consists of two parts. The first part describes how to create the model with a detailed geometry. The second part shows how to define a homogeneous model for porous media using an effective diffusivity calculated using the detailed model from the first part.

Introduction

Transport through porous structures is usually treated using simplified homogeneous models with effective transport properties. This is in most cases a necessity, since the typical dimensions of the pores and particles making up the porous structure are several orders of magnitude smaller than the size of the subdomain that is to be modeled.

However, it might be interesting to investigate the assumptions and simplifications done when homogenizing a porous structure by comparing a homogeneous model with a model defined using the detailed structure.

The artificial porous structure used in this model is depicted in Figure 4-1 below.

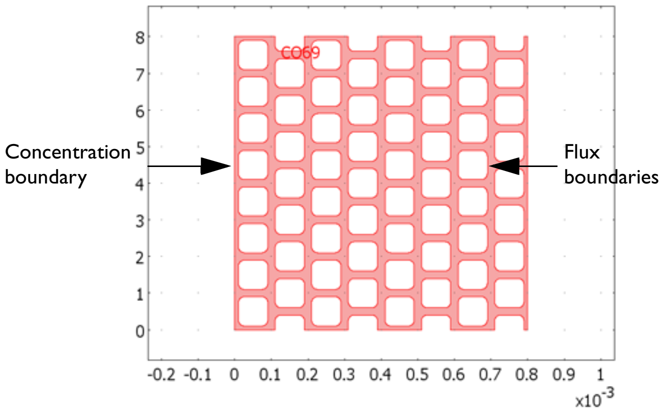


Figure 4-1: Artificial porous structure. The domain colored in red is accessible for diffusion.

Model Definition

The model equation in the modeled domain shown in Figure 4-1 is the time-dependent equation

$$\frac{\partial c}{\partial t} + \nabla \cdot (-D\nabla c) = 0$$

where c denotes concentration (mol/m^3 using SI units) and D the diffusion coefficient (m^2/s) of the solute.

The boundary conditions are of three different types. The so-called concentration boundary condition is set at the left vertical boundary in Figure 4-1 and is expressed as

$$c = c_0$$

where c_0 is a given concentration.

The right vertical boundary in Figure 4-1 is set according to

$$(-D\nabla c) \cdot \mathbf{n} = k_m(c - c_1)$$

where k_m is the mass transfer coefficient (m/s) and c_1 the concentration in a bulk solution outside of the porous structure.

All other boundaries are insulating boundaries according to

$$(-D\nabla c)\cdot\mathbf{n} = 0$$

The initial condition is given by a bell-shaped profile along the x-axis with its maximum at $x = 0$ and a corresponding value of $c = c_0$:

$$c(t_0) = c_0\exp(-ax^2)$$

The solution is assumed to be a gaseous solution with a content of solute of 3 mol/m^3 at the so-called concentration boundary. The diffusion coefficient is set to $1\cdot10^{-5}$ m^2/s.

The second part of this exercise uses a homogenized 1D model geometry with effective transport properties and an average porosity. The model equation then becomes:

$$\varepsilon\frac{\partial c}{\partial t} + \nabla\cdot(-D^{\mathrm{eff}}\nabla c) = 0$$

where ε denotes the average porosity and D^{eff} the effective diffusivity. The porosity and effective diffusivity are calculated from the results of the detailed structure, see the section below. The boundary conditions are the concentration and flux conditions according to the above equations.

Results and Discussion

The simulations are run for $t = 0$ to 0.1 s, when the simulation reaches steady state. Figure 4-2 below shows the concentration profile after 0.05 s in the porous structure.

Already at this stage the concentration has almost reached steady state, which is visible by the nearly linear concentration profile across the structure.

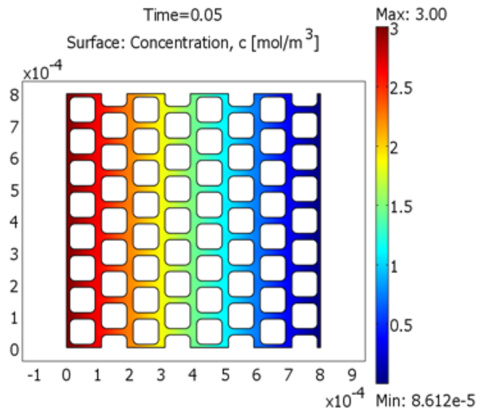


Figure 4-2: Concentration profile in the modeled artificial porous structure at t = 0.05 s.

When modeling porous media, the exact concentration in the pore structure is not the most important issue because the description of the structure is homogenized and not detailed as in Figure 4-2. The most interesting issue is then the description of the flux. To calculate the average flux, integrate over the flux boundary and divide by its length, L_0, which yields the following expression:

$$N_{\text{average}} = \frac{1}{L_0}\int_0^{L_0} k_m(c - c_1)\,dS.$$

Figure 4-3 shows the value of this integral as a function of time. If you let the process reach steady-state, the average flux becomes $8.051\cdot10^{-3}$ mol·m^2/s. Considering the almost linear profile across the structure, it is natural to replace the porous structure with a 1D homogenized structure along the x-axis. It is then possible to calculate the effective diffusivity according to the following:

$$D^{\text{eff}}\frac{(c_0 - c_{\text{out}})}{L_1} = N_{\text{average}}$$

where c_{out} is the average concentration (mol/m^3) at the flux boundary, and L_1 is the length of the geometry along the x-axis. The average concentration is obtained by integrating according to the expression below:

$$c_{\text{out}} = \frac{1}{L_0}\int_0^{L_0} c\,dS.$$

This gives and average concentration $c_{\text{out}} = 1.61 \cdot 10^{-3}$ mol/m^3 and using $L_1 = 8 \cdot 10^{-4}$ m gives the effective diffusivity according to the following:

$$D^{\text{eff}} = \frac{8.051 \cdot 10^{-3} \times 8.0 \cdot 10^{-4}}{(3 - 1.61 \cdot 10^{-3})}$$

which yields a value for the effective diffusivity of $2.15 \cdot 10^{-6}$ m^2/s compared to the "free" diffusivity of $1 \cdot 10^{-5}$ m^2/s. The effective and "free" diffusivities are usually related according to the equation

$$D^{\text{eff}} = D\frac{\varepsilon}{\tau}$$

where ε is the porosity of the structure and τ the so-called tortuosity, which is a measure of the actual length per unit effective length a molecule has to diffuse in a porous structure. To calculate the porosity of the modeled structure, you integrate the value 1 over the structure and then divide this by the length and width of the structure:

$$\varepsilon = \frac{1}{L_0 L_1}\int_0^{L_1}\int_0^{L_0} 1\,dx\,dy$$

resulting in a value of 0.382. The value of τ can then be calculated to 1.78. In addition, the tortuosity is usually expressed as a power of the porosity, resulting in an expression for the effective diffusivity according to

$$D^{\text{eff}} = D\varepsilon^p$$

If you use the calculated values for porosity and effective diffusivity, the value for p is 1.60. The experimental values for p for porous structures used for transport in catalysts, soils, and other porous structures is usually in the range 1.5–2.

Using the value of the effective diffusivity, a simple homogenized 1D model provides the possibility to compare the value of the flux using a homogenized model to the value using the detailed 2D structure. Figure 4-3 shows that there is an excellent agreement between the model using a detailed geometry and the homogenized model.

The difference in the time-dependent flux is hardly visible between the two cases in the graph.

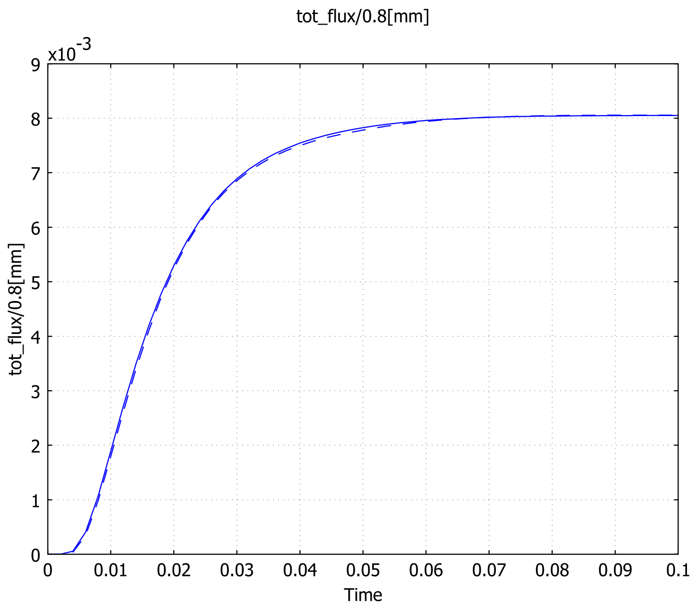


Figure 4-3: Average flux at the flux boundary in the detailed 2D model (solid line) and the 1D homogenized approximation (dashed line).

Modeling in COMSOL Multiphysics

Both models described above are straightforward to define in COMSOL Multiphysics. One feature that is of great use in this model is the ability to define integration coupling variables to automatically generate the values of the integrals needed to evaluate the results of the model. The definition of these integrals is described in detail in the step-by-step instructions below.

Model Library path: `COMSOL_Multiphysics/Diffusion/effective_diffusivity`

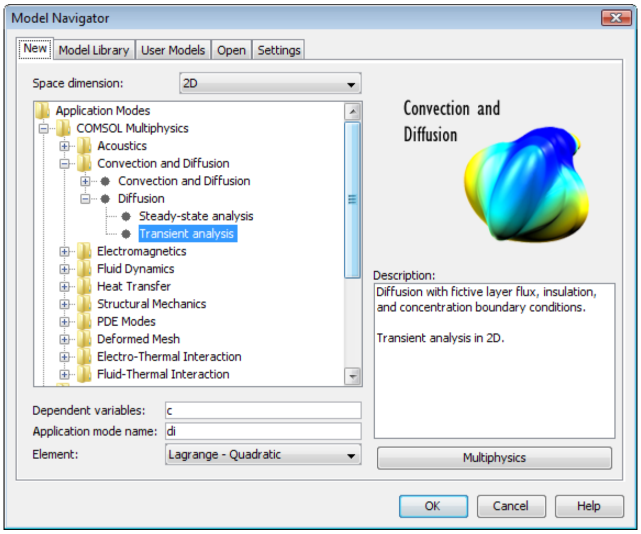
Modeling Using the Graphical User Interface

You can start by setting up the first and most extensive part of the exercise, which is the 2D model of the detailed porous structure.

1 Double-click on the COMSOL Multiphysics icon on your desktop to open the **Model Navigator**.

2 Select **2D** from the **Space dimension** list.

3 Select **COMSOL Multiphysics>Convection and Diffusion>Diffusion>Transient analysis** as in the figure below.



Space dimension: 2D

Dependent variables: c

Application mode name: di

Element: Lagrange - Quadratic

Description:
Diffusion with fictive layer flux, insulation, and concentration boundary conditions.

Transient analysis in 2D.

4 Click **OK**.

OPTIONS AND SETTINGS

Define the constants required as input to the model.

1 Select **Constants** in the **Options** menu.

2 Define the concentration `c0` in the **Name** column in the **Constants** dialog box.

3 Type `3[mol/m^3]` in the corresponding **Expression** column.

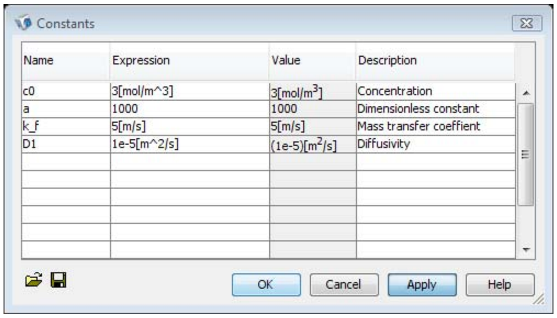
4 Type `Concentration` in the corresponding **Description** column.

The **Name** column defines the name that you have to use to refer back to this constant, the **Expression** column is used to calculate its **Value**, and the **Description** column is used to make notes regarding this constant. Any constant can be a function of the other constants, which is the reason why the **Expression** column is not always identical to the **Value** column.

5 Below the above constant, type `a` in the **Name** column, `1000` in the **Expression** column, and `Dimensionless constant` in the **Description** column.

6 Continue generating the constant list by typing `k_f` in the **Name** column, `5[m/s]` in the **Expression** column, and `Mass transfer coefficient` in the **Description** column.

7 Type `D1` in the **Name** column, `1e-5[m^2/s]` in the **Expression** column, and `Diffusivity` in the **Description** column. The constant dialog box should look according to the figure below. You can now save this list for later access in the 1D model in the second part of the exercise.



Name	Expression	Value	Description
c0	3[mol/m^3]	$3[mol/m^3]$	Concentration
a	1000	1000	Dimensionless constant
k_f	5[m/s]	5[m/s]	Mass transfer coeffient
D1	1e-5[m^2/s]	$(1e-5)[m^2/s]$	Diffusivity

8 Click **Apply**.

9 Click the **Export Variables To File** button in the **Constants** dialog box (floppy disk symbol).

10 Save the data file in your working directory as `diffusion.txt`.

11 Click **Save** in the **Export Variables** dialog box.

12 Click **OK** in the **Constants** dialog box.

Continue with the functions.

1 In the **Options** menu, choose **Expressions>Scalar Expressions**.

2 Define a concentration function `c1` in the **Name** column in the **Scalar Expressions** dialog box.

3 Type `c0*exp(a*(-(x/4e-4[m])^2))` in the corresponding **Expression** column.

4 Type `Concentration` in the corresponding **Description** column.

5 Click **Apply**. You can now save the expression for use in the 1D model.

6 Click the **Export Variables To File** button in the **Scalar Expressions** dialog box (floppy disk symbol).

7 Save the data file in your working directory as `diffusion1.txt`.

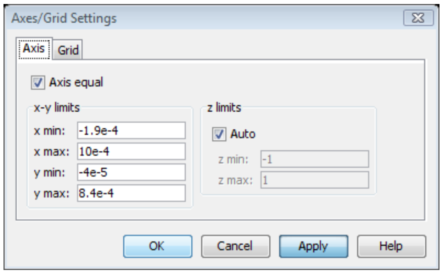
8 Click **Save** in the **Export Variables** dialog box.

9 Click **OK** in the **Scalar Expressions** dialog box.

The last step in the **Options** menu is to set the size of the drawing table.

1 Select **Axes/Grid Settings** in the **Options** menu.

2 Set the minimum and maximum values for the *x*- and *y*-axes according to the figure below. You can do this by clicking on the edit fields and typing the corresponding values.



[✓] Axis equal

x-y limits

x min: -1.9e-4
x max: 10e-4
y min: -4e-5
y max: 8.4e-4

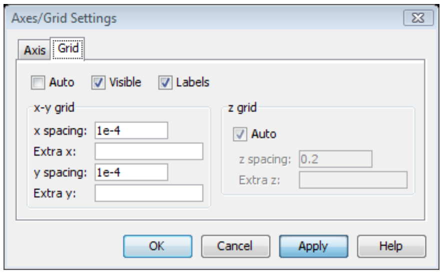
z limits

[✓] Auto

z min: -1
z max: 1

3 Click the **Grid** tab and clear the **Auto** check box.

4 Set the **x spacing** and **y spacing** grid lines as in the figure below.



[ ] Auto [✓] Visible [✓] Labels

x-y grid

x spacing: 1e-4
Extra x:
y spacing: 1e-4
Extra y:

z grid

[✓] Auto

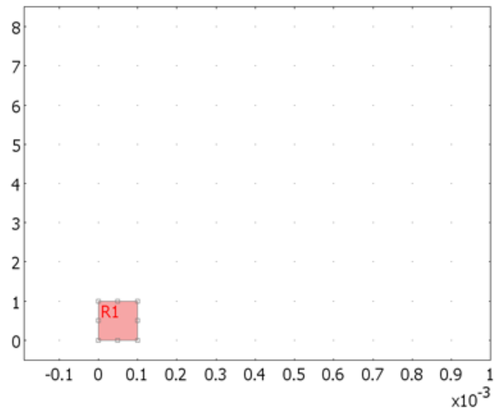
z spacing: 0.2
Extra z:

5 Click **OK**.

GEOMETRY MODELING

1 Click the **Rectangle/Square** button on the Draw toolbar.

2 Use the mouse to drag from the (x, y) coordinates $(0, 0)$ to (1e-4, 1e-4) to create a square with one corner at the origin and one corner across the diagonal from origin at (1e-4, 1e-4); see the figure below.



3 Click the **Fillet/Chamfer** button on the Draw toolbar.

4 Select Vertex 1 in the **Vertex selection** list.

5 Type `2e-5` in the **Radius** edit field.

6 Click **Apply**. The rectangle R1 now changes name to CO1 (composite object 1).

7 Select Vertex 2 in the **Vertex selection** list.

8 Click **Apply**. The name is now changed to CO2.

9 Select Vertex 5 in the **Vertex selection** list.

10 Click **Apply**. The name is now changed back to CO1 (this since the name is not taken anymore).

11 Select Vertex 7 in the **Vertex selection** list.

12 Click **Apply**. The name is now changed again back to CO2.

13 Click **OK**.

You should now have the rectangle according to the figure above but now with rounded corners.

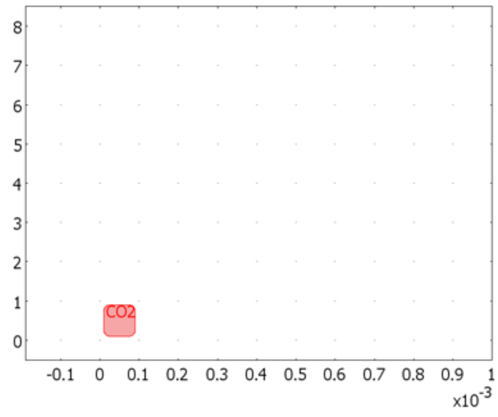
1 Click the **Scale** button (with CO2 selected; that is, colored red).

2 Type `0.8` in both the **x** and **y** edit fields in the **Scale factor** area, then click **OK**.

3 Click the **Move** button on the Draw toolbar.

4 Type `1e-5` in both the **x** and **y Displacement** edit fields.

The rounded square should now be positioned according to the figure below.



You can now copy and paste to create the porous structure.

1 Click the **Array** button on the Draw toolbar.

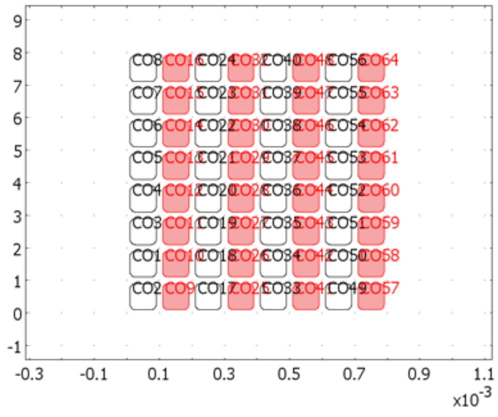
2 Enter `1e-4` in the **x** and **y** edit fields in the **Displacement** area.

3 In the **Array size** edit fields enter `8` for both **x** and **y**.

4 Click **OK**.

You now have to select each second column of objects and displace them.

1 Press the Ctrl key and drag the mouse over the objects in each second column to obtain the selection according to the figure below. You can use the rubber band functionality by dragging the mouse. You keep the selection by pressing the Ctrl key.

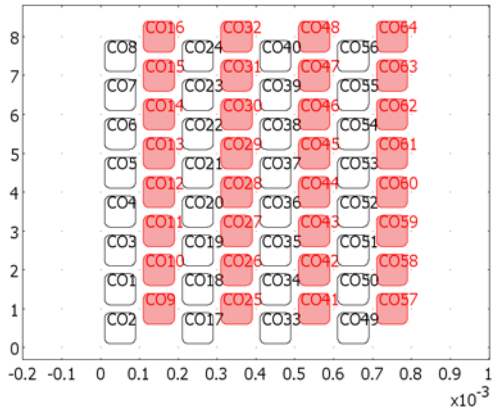


2 Click the **Move** button on the Draw toolbar.

3 Type 5e-5 in the **y** edit field in the **Displacement** area.

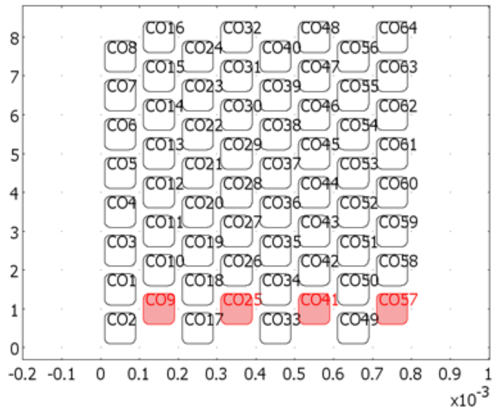
4 Click **OK**.

5 Click **Zoom Extents**. Your graphical user interface should look as in the figure below.



6 Select **Deselect All** in the **Edit** menu.

7 Press the Ctrl key and click the objects CO9, CO25, CO41, CO57 to obtain the selection according to the following figure.

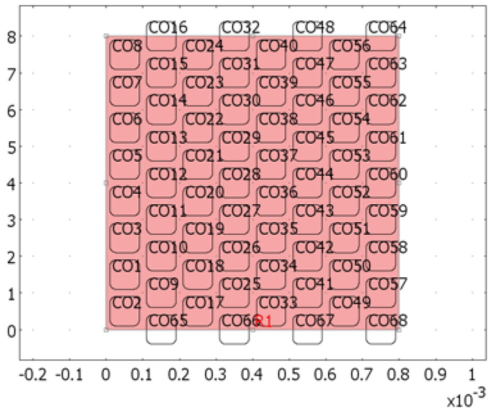


8 Press Ctrl+C to copy and Ctrl+V to paste the selected objects.

9 In the **Paste** dialog box, type -1e-4 in the **y** edit field in the **Displacements** area. Click **OK**.

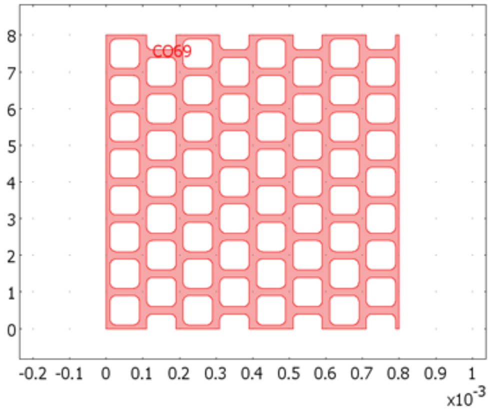
10 Click the **Zoom Extents** button on the Main toolbar.

11 Click the **Rectangle/Square** button and drag the mouse from the coordinates (x, y) (0, 0) to $(8{\cdot}10^{-4}, 8{\cdot}10^{-4})$ to create a square with one corner at origin and the corner across the diagonal from the origin at $(8{\cdot}10^{-4}, 8{\cdot}10^{-4})$. The graphical user interface should look according to the figure below.



12 Press Ctrl+A to select all objects.

13 Click the **Difference** button. The geometry should be according to the figure below.



This finalizes the geometry section for the first part of this exercise. You can now continue with the properties and boundary conditions.

PHYSICS SETTINGS

Subdomain Settings

1 Select **Subdomain Settings** from the **Physics** menu.

2 Select Subdomain 1 from the **Subdomain selection** list.

3 Type `D1` in the **Diffusion coefficient** edit field. Note that you have defined `D1` in the **Constants** dialog box in the **Options** menu.

4 Click the **Init** tab to set the initial condition.

5 Type `c1` in the **Concentration, c** edit field (**c(t_0)**). Note that you have defined `c1` as a scalar expression variable.

6 Click **OK**.

Boundary Conditions

1 Select **Boundary Settings** from the **Physics** menu.

2 Select Boundary 1 in the **Boundary selection** list.

3 Select **Concentration** from the **Boundary condition** list.

4 Type `c0` in the **Concentration** edit field.

5 Select Boundary 276 from the **Boundary selection** list.

6 Select **Flux** from the **Boundary condition** list.

7 Type `k_f` in the **Mass transfer coefficient** edit field. The bulk concentration in the **Bulk concentration** edit field should be `0`.

8 Click **OK**.

All other boundaries, except for the two boundaries that you have altered above, should remain at the default Insulation/Symmetry boundary condition.

MESH GENERATION

Click the **Initialize Mesh** button on the Main toolbar.

COMPUTING THE SOLUTION

1 Click the **Solver Parameters** button on the Main toolbar.

2 Type `0:2e-3:0.1` in the **Times** edit field to define the **Time stepping**. This gives an integration from 0 to 0.1 s with 2 ms increments between each output.

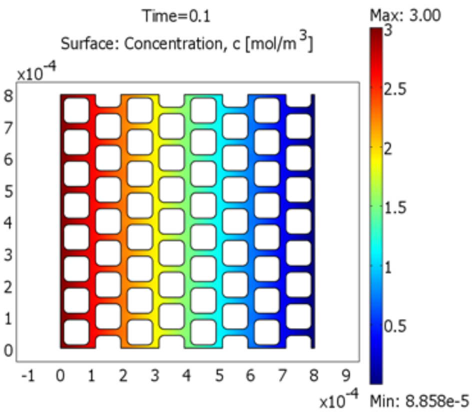
3 Select **Direct (SPOOLES)** from the **Linear system solver** list. The SPOOLES solver makes use of the symmetry in the diffusion equation, which the solver detects automatically, thus saving memory.

4 Click **OK**.

5 Click the **Solve** button on the Main toolbar.

POSTPROCESSING AND VISUALIZATION

The default plot gives the concentration at the final time step 0.1 s; see the figure below.



You can now continue generating the integral expressions needed to evaluate the average flux. To plot the integrals as a function of time, you have to create an integration coupling variable.

1 In the **Options** menu, select **Integration Coupling Variables>Boundary Variables**.
2 Select Boundary 276 in the **Boundary selection** list.
3 Type `tot_flux` in the **Name** column.
4 Type `k_f*c` in the **Expression** column.
5 Clear the **Global destination** check box.
6 Click the **Destination** tab.
7 Check the box for Point 532 in the **Point selection** list.
8 Click **OK**.

You have now defined the integration variable. However, the value has not been evaluated yet. To do that, you do not have to run the solution process; instead just update the solution:

Select **Update Model** from the **Solve** menu.

This should evaluate the integral at all output times. You can now plot the value of the average flux:

1 Select **Domain Plot Parameters** from the **Postprocessing** menu.

2 Click the **Point** tab. Select Point 532 in the **Point selection** list.

3 Type `tot_flux/0.8[mm]` in the **Expression** edit field to plot the average flux (0.8 mm is the length of the integration boundary).

4 Click **OK**.

This generates the solid line in Figure 4-3.

To get the porosity of the domain for the 1D model, do a subdomain integration:

1 Select **Subdomain Integration** from the **Postprocessing** menu.

2 Select Subdomain 1 in the **Subdomain selection** list.

3 Type `1/(0.8[mm])^2` in the **Expression** edit field.

4 Click **OK**. The value 0.381976 should be visible in the message window.

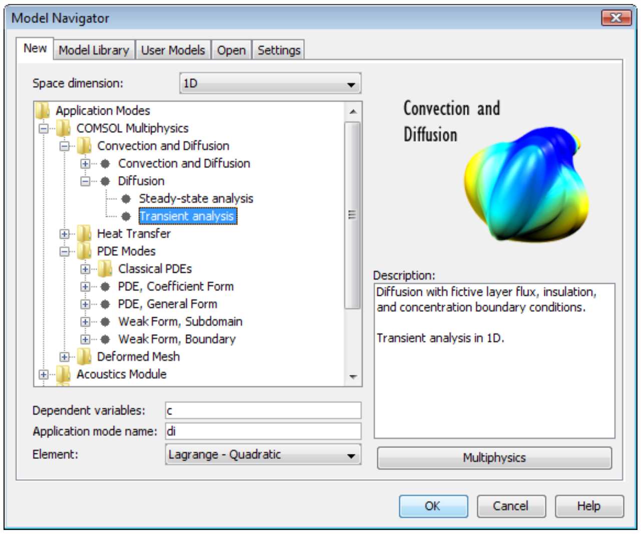
5 Save the model in your working directory.

You can now continue with the 1D model. Note that the figure window denoted **Figure 1**, showing the average flux of species, should still be open.

MODEL NAVIGATOR

1 Click the **New** button. Select **1D** from the **Space dimension** list.

2 Select **COMSOL Multiphysics>Convection and Diffusion>Diffusion>Transient analysis** from the **Application Modes** tree; see the figure below.



Space dimension: 1D

Description:
Diffusion with fictive layer flux, insulation, and concentration boundary conditions.

Transient analysis in 1D.

Dependent variables: c
Application mode name: di
Element: Lagrange - Quadratic

3 Click **OK**.

OPTIONS AND SETTINGS

You can now load the constants from the 2D model above.

1 Select **Constants** from the **Options** menu.
2 Click the **Import Variables From File** button in the lower left corner of the **Constants** dialog box (open-catalog symbol).
3 Select the file `diffusion.txt` that you have previously created in your working directory in the **Import Variables** dialog box.
4 Click **Open**.
5 Define a new constant in the **Constants** dialog box by typing `epsilon` in the **Name** column, `0.381976` in the **Expression** column, and `Porosity` in the **Description** column.
6 Change the diffusivity `D1` to `2.15e-6` in the corresponding **Expression** field.
7 Click **OK** in the **Constants** dialog box.

You can continue by loading the scalar expression from the 2D model.

1 Select **Expressions>Scalar Expressions** from the **Options** menu.
2 Click the **Import Variables From File** button in the lower left corner of the **Scalar Expressions** dialog box (open-catalog symbol).
3 Select the file `diffusion1.txt` that you have previously created in your working directory in the **Import Variables** dialog box.
4 Click **Open**.
5 Click **OK** in the **Scalar Expressions** dialog box.

GEOMETRY MODELING

1 Select **Specify Objects**, **Line** from the **Draw** menu.
2 Type `0 8e-4` in the **Coordinate**s, **x** edit field to create a line from $x = 0$ to $x = 8\cdot10^{-4}$.
3 Click the **Zoom Extents** button on the Main toolbar.

PHYSICS

Subdomain Settings

1 Select **Subdomain Settings** from the **Physics** menu.
2 Select Subdomain 1 from the **Subdomain selection** list.
3 Type `epsilon` in the **Time-scaling coefficient** edit field.
4 Type `D1` in the **Diffusion coefficient** edit field.

5 Click the **Init** tab.

6 Type `c1` in the **Concentration, c** edit field (**c(t_0)**). This sets the initial condition.

7 Click **OK**.

Boundary Settings

1 Select **Boundary Settings** from the **Physics** menu.

2 Select Boundary 1 in the **Boundary selection** list.

3 Select **Concentration** in the **Boundary condition** drop-down list.

4 Type `c0` in the **Concentration** edit field.

5 Select Boundary 2 in the **Boundary selection** list.

6 Select **Flux** from the **Boundary condition** list.

7 Type `k_f` in the **Mass transfer coefficient** edit field.

8 Click **OK**.

MESH GENERATION

Click the **Refine Mesh** button on the Main toolbar to create a fine mesh.

COMPUTING THE SOLUTION

1 Click the **Solver Parameters** button on the Main toolbar.

2 Type `0:2e-3:0.1` in the **Times** edit field to define the **Time stepping**.

3 Click **OK**.

4 Click the **Solve** button on the Main toolbar.

POSTPROCESSING AND VISUALIZATION

1 Select **Domain Plot Parameters** in the **Postprocessing** menu.

2 Click the **Point** tab. Select Point 2 in the **Point selection** list.

3 Type `k_f*c` in the **Expression** edit field.

4 Click the **Line Settings** button.

5 Select **Dashed line** from the **Line style** list, then click **OK**.

6 Click the **General** tab. Select the **Keep current plot** check box.

7 Select **Figure 1** in the **Plot in** list, then click **OK**.

You should now get a plot according to Figure 4-3 above.

5

Electromagnetics

This chapter explains the application modes in COMSOL Multiphysics for electromagnetics and how to use them for electromagnetic field simulations. Specifically, it takes a detailed look at the Conductive Media DC application mode, the Electrostatics application mode, the Magnetostatics application mode, and the AC Power Electromagnetics application mode. It concludes by presenting three step-by-step examples using the Conductive Media application mode DC to model a copper plate, the Electrostatics application mode to model an electric sensor, and the Magnetostatics application mode to model a permanent magnet.

The Electromagnetics Application Modes

For simulating electromagnetic fields, COMSOL Multiphysics offers four application modes.

The first three perform static simulations:

- Electrostatics
- Conductive media DC
- Magnetostatics

With static problems you solve for either electric properties, as is the case for the first two modes, or magnetic properties, as in magnetostatics.

In time-varying electromagnetic fields the electric and magnetic quantities are coupled, so you simulate them with the fourth application modes of interest here: AC Power Electromagnetics.

This section of the manual begins with a brief introduction to electromagnetics and a definition of the electromagnetic quantities. Then it discusses each of the four application modes in detail.

Note: The optional AC/DC Module contains specialized and extended application modes and models for electromagnetic simulations, for example, for computations of inductors and capacitors. The optional RF Module includes application modes for wave-propagation simulations that are especially useful in microwave engineering and photonics.

Fundamentals of Electromagnetics

The problem of electromagnetic analysis on a macroscopic level is that of solving *Maxwell's equations* subject to certain boundary conditions. Maxwell's equations are a set of equations, written in differential or integral form, stating the relationships between the fundamental electromagnetic quantities. These quantities are:

- The *electric field intensity*, $\mathbf{E}$
- The *electric displacement* or *electric flux density*, $\mathbf{D}$
- The *magnetic field intensity*, $\mathbf{H}$
- The *magnetic flux density*, $\mathbf{B}$
- The *current density*, $\mathbf{J}$
- The *electric charge density*, ρ

You can formulate the equations in differential or integral form. This discussion presents them in differential form because it leads to differential equations that the finite element method can be handle.

For general time-varying fields, Maxwell's equations are

$$\begin{aligned}\nabla\times\mathbf{H} &= \mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}\\ \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}\\ \nabla\cdot\mathbf{D} &= \rho\\ \nabla\cdot\mathbf{B} &= 0\end{aligned}$$

The first two equations are also referred to as *Maxwell-Ampère's law* and *Faraday's law*, respectively. The last two are forms of *Gauss' law* in the electric and magnetic form, respectively.

Another fundamental relationship is the *equation of continuity*:

$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}$$

Out of these five equations only three are independent. The first two combined with either the electric form of Gauss' law or the equation of continuity form an independent system.

CONSTITUTIVE RELATIONSHIPS

To obtain a closed system, you need the *constitutive relationships* describing the macroscopic properties of the medium. They are

$$\begin{aligned}\mathbf{D} &= \varepsilon_0\mathbf{E}+\mathbf{P}\\ \mathbf{B} &= \mu_0(\mathbf{H}+\mathbf{M})\\ \mathbf{J} &= \sigma\mathbf{E}\end{aligned} \tag{5-1}$$

where ε_0 is the *permittivity of vacuum,* μ_0 is the *permeability of vacuum,* and σ is the *electric conductivity.* In the SI system the permeability of a vacuum is $4\pi\cdot10^{-7}$ H/m. The velocity of an electromagnetic wave in a vacuum is given as c_0, and you can derive the permittivity of a vacuum from the relationship

$$\varepsilon_0 = \frac{1}{c_0^2\mu_0} = 8.854\cdot10^{-12}\ \mathrm{F/m} \approx \frac{1}{36\pi}\cdot10^{-9}\ \mathrm{F/m}$$

The *electric polarization vector* **P** describes how a material is polarized when an electric field **E** is present. It can be interpreted as the volume density of electric dipole moments. **P** is generally a function of **E**. Some materials can have a nonzero **P** in the absence of an electric field.

The *magnetization vector* **M** similarly describes how a material is magnetized when a magnetic field **H** is present. It can be interpreted as the volume density of magnetic dipole moments. **M** is generally a function of **H**. One use of the magnetization vector is to describe permanent magnets, which have a nonzero **M** when no magnetic field is present.

For linear materials the polarization is directly proportional to the electric field, $\mathbf{P}=\varepsilon_0\chi_e\mathbf{E}$, where χ_e is the *electric susceptibility.* Similarly, in linear materials the magnetization is directly proportional to the magnetic field, $\mathbf{M}=\chi_m\mathbf{H}$, where χ_m is the *magnetic susceptibility.* For such materials the constitutive relations are

$$\begin{aligned}\mathbf{D} &= \varepsilon_0(1+\chi_e)\mathbf{E} = \varepsilon_0\varepsilon_r\mathbf{E} = \varepsilon\mathbf{E}\\ \mathbf{B} &= \mu_0(1+\chi_m)\mathbf{H} = \mu_0\mu_r\mathbf{H} = \mu\mathbf{H}\end{aligned}$$

where ε_r is the material's *relative permittivity*, and μ_r is its *relative permeability.* Usually these are scalar properties but can, in the general case, be 3-by-3 tensors when the material is anisotropic. The properties ε and μ (without subscripts) are the material's *permittivity* and *permeability.*

Generalized Constitutive Relationships

For nonlinear materials, a generalized form of the constitutive relationships is useful. The relationship used for electric fields is

$$\mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E} + \mathbf{D}_r$$

The field $\mathbf{D}_r$ is the *remanent displacement*, which is the displacement when no electric field is present.

Similarly, a generalized form of the constitutive relationship for the magnetic field is

$$\mathbf{B} = \mu_0\mu_r\mathbf{H} + \mathbf{B}_r$$

where $\mathbf{B}_r$ is the *remanent magnetic flux density*, which is the magnetic flux density when no magnetic field is present.

You can generalize the third line in Equation 5-1 by introducing an externally generated current $\mathbf{J}^e$. This relationship is then

$$\mathbf{J} = \sigma\mathbf{E} + \mathbf{J}^e$$

POTENTIALS

Under certain circumstances it can be helpful to formulate a problem in terms of the *electric scalar potential* V and *magnetic vector potential* $\mathbf{A}$. They are given by the equalities

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} \\ \mathbf{E} &= -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \end{aligned}$$

which are direct consequences of the magnetic case of Gauss' law and Faraday's law, respectively.

MATERIAL PROPERTIES

This discussion has so far only formally introduced the constitutive relationships. These seemingly simple relationships can be quite complicated at times. In fact, these relationships require some special considerations when working with four main groups of materials:

- Inhomogeneous materials
- Anisotropic materials

- Nonlinear materials
- Dispersive materials

A material can belong to one or more of these groups.

Inhomogeneous materials are the least complicated. An inhomogeneous medium is one in which the constitutive parameters vary with the space coordinates so that different field properties prevail at different parts of the material structure.

For anisotropic materials the field relationships at any point differ for different directions of propagation. This means that a 3x3 tensor is necessary to properly define the constitutive relationships. If this tensor is symmetric, the material is often referred to as *reciprocal*. In these cases you can rotate the coordinate system such that a diagonal matrix results. If two of the diagonal entries are equal, the material is *uniaxially anisotropic*; if none of the elements have the same value, the material is *biaxially anisotropic* (Ref. 2). You need anisotropic parameters, for instance, to examine permittivity in crystals (Ref. 2) and when working with conductivity in solenoids.

In some nonlinear materials the permittivity or permeability depend on the intensity of the electromagnetic field. Nonlinearity also includes hysteresis effects where not only the existing field intensities influence a material's physical properties but the history of the field distribution also plays a role.

Finally, dispersion describes changes in a wave's velocity with wavelength. In the frequency domain you can express dispersion with a frequency dependence of the constitutive laws.

BOUNDARY AND INTERFACE CONDITIONS

To get a full description of an electromagnetic problem, you must also specify boundary conditions at material interfaces and physical boundaries. At interfaces between two media, you can mathematically express the boundary conditions as

$$\begin{aligned}
\mathbf{n}_2 \times (\mathbf{E}_1 - \mathbf{E}_2) &= \mathbf{0} \\
\mathbf{n}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) &= \rho_s \\
\mathbf{n}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) &= \mathbf{J}_s \\
\mathbf{n}_2 \cdot (\mathbf{B}_1 - \mathbf{B}_2) &= 0
\end{aligned}$$

where ρ_s and $\mathbf{J}_s$ denote the *surface charge density* and *surface current density*, respectively, and $\mathbf{n}_2$ is the outward normal from medium 2. Of these four equations, only two are independent. This is an overdetermined system of equations, so you would like to reduce it. First select either equation one or equation four. Then select either equation two or equation three. Together these selections form a set of two independent conditions.

From these relationships, you can derive the interface condition for the current density,

$$\mathbf{n}_2 \cdot (\mathbf{J}_1 - \mathbf{J}_2) = -\frac{\partial \rho_s}{\partial t}$$

Interface Between a Dielectric and a Perfect Conductor

A perfect conductor has infinite electrical conductivity and as such has no internal electric field. Otherwise it would produce an infinite current density according to the third fundamental constitutive relationship. At an interface between a dielectric and a perfect conductor, the boundary conditions for the **E** and **D** fields are simplified. Assume that subscript 1 corresponds to a perfect conductor; then $\mathbf{D}_1 = \mathbf{0}$ and $\mathbf{E}_1 = \mathbf{0}$ in the relationships just given. If, in addition, you are dealing with a time-varying case, then $\mathbf{B}_1 = \mathbf{0}$ and $\mathbf{H}_1 = \mathbf{0}$, as well, as a consequence of Maxwell's equations. The result is the following set of boundary conditions for the fields in the dielectric medium for the time-varying case:

$$\begin{aligned} -\mathbf{n}_2 \times \mathbf{E}_2 &= 0 \\ -\mathbf{n}_2 \times \mathbf{H}_2 &= \mathbf{J}_s \\ -\mathbf{n}_2 \cdot \mathbf{D}_2 &= \rho_s \\ -\mathbf{n}_2 \cdot \mathbf{B}_2 &= 0 \end{aligned}$$

Electromagnetic Force for Particle Tracing

The electromagnetic force is available for particle tracing plots in the Conductive Media DC application mode and the Electrostatics application mode in the COMSOL Multiphysics products.

The following expression is used to describe the electromagnetic force on a particle with the charge q moving through the electric and magnetic fields with the velocity $\mathbf{v}$:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

USING THE ELECTROMAGNETIC FORCE FOR PARTICLE TRACING

The electromagnetic force is the default selection in the **Predefined forces** list on the **Particle Tracing** page.

There is one parameter in this force expression: the particle charge, q, which you define by clicking the **Parameters** button. The default value of q is the elementary charge:

PARAMETER NAME IN EQUATION	DESCRIPTION	DEFAULT VALUE	DEFAULT VARIABLE NAME
q	particle charge	$1.602 \cdot 10^{-19}$ C	partq

Note: The default setting works for models that use SI units.

The Conductive Media DC Application Mode

Electrolysis and the computation of resistances of grounding plates involves a medium with electric conductivity σ and a steady current.

You can use the *Conductive Media DC* application mode for 3D, 2D in-plane, and 2D axisymmetric models.

PDE Formulation

When handling conductive media you must consider the equation of continuity. In a stationary coordinate system, the point form of *Ohm's law* states that

$$\mathbf{J} = \sigma\mathbf{E} + \mathbf{J}^e$$

where $\mathbf{J}^e$ is an externally generated current density. The static form of the equation of continuity then gives

$$\nabla \cdot \mathbf{J} = -\nabla \cdot (\sigma\nabla V - \mathbf{J}^e) = 0$$

To handle current sources, generalize the equation to

$$-\nabla \cdot (\sigma\nabla V - \mathbf{J}^e) = Q_j$$

The in-plane *Conductive Media DC* application mode assumes that your model has a symmetry where the electric potential varies only in the x and y directions and is constant in the z direction. This implies that the electric field, $\mathbf{E}$, is tangential to the xy-plane. The application mode solves the following equation where d is the thickness in the z direction:

$$-\nabla \cdot d(\sigma\nabla V - \mathbf{J}^e) = dQ_j$$

The axisymmetric *Conductive Media DC* application mode considers the situation where the fields and geometry are axially symmetric. In this case the electric potential is constant in the φ direction, which implies that the electric field is tangential to the rz-plane.

Writing the generalized equation for Q_j in cylindrical coordinates and multiplying it by r to avoid singularities at $r = 0$ results in

$$-\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial z} \end{bmatrix}^T \cdot \left(r\sigma \begin{bmatrix} \frac{\partial V}{\partial r} \\ \frac{\partial V}{\partial z} \end{bmatrix} - r\mathbf{J}^e \right) = rQ_j$$

SPECIFYING THE CONDUCTIVITY

You can provide the conductivity in three different ways:

- Isotropic conductivity: a scalar number or expression
- Anisotropic conductivity: several components of a conductivity tensor to define an anisotropic material
- Temperature-dependent conductivity (which occurs in, for example, Joule heating, which is also called resistive heating). In this case the following equation describes the conductivity:

$$\sigma = \frac{1}{\rho_0(1 + \alpha(T - T_0))}$$

where ρ_0 is the resistivity at the reference temperature T_0. α is the temperature coefficient of resistivity, which describes how the resistivity varies with temperature. T is the current temperature, which can be a value that you specify or the temperature from a heat transfer application mode (in the Joule Heating predefined multiphysics coupling, this is the default setting).

Boundary Conditions

The relevant interface condition at interfaces between different media for this application mode is

$$\mathbf{n}_2 \cdot (\mathbf{J}_1 - \mathbf{J}_2) = 0$$

This is fulfilled by the natural boundary condition

$$\mathbf{n} \cdot [(\sigma\nabla V - \mathbf{J}^e)_1 - (\sigma\nabla V - \mathbf{J}^e)_2] = -\mathbf{n} \cdot (\mathbf{J}_1 - \mathbf{J}_2) = 0$$

CURRENT FLOW

The current-flow boundary condition

$$\mathbf{n} \cdot \mathbf{J} = \mathbf{n} \cdot \mathbf{J}_0$$

specifies the normal component of the current density for the current flowing across the boundary.

INWARD CURRENT FLOW

The inward-current flow boundary condition

$$-\mathbf{n} \cdot \mathbf{J} = J_n$$

is similar to the previous current-flow boundary condition except this case specifies the normal component of the current density rather than the complete vector. When the normal component J_n is positive then the current flows inwards towards the boundary.

DISTRIBUTED RESISTANCE

You can use the distributed resistance boundary condition

$$\mathbf{n} \cdot \mathbf{J} = \frac{\sigma}{d}(V - V_{\text{ref}}), \qquad \mathbf{n} \cdot (\mathbf{J}_1 - \mathbf{J}_2) = \frac{\sigma}{d}(V - V_{\text{ref}})$$

to model a thin sheet of a resistive material. The sheet has the thickness d and is connected to the potential V_{ref}.

ELECTRIC INSULATION

The electric-insulation boundary condition

$$\mathbf{n} \cdot \mathbf{J} = 0$$

specifies that no current flows across the boundary.

This boundary condition is also applicable at symmetric boundaries where the potential is known to be symmetric with respect to the boundary.

ELECTRIC POTENTIAL

The electric-potential boundary condition

$$V = V_0$$

specifies the voltage at a boundary. Because you are solving for the potential in this application mode, you generally define the value of the potential at some boundary in the geometry.

GROUND

The ground boundary condition

$$V = 0$$

is a special case of the previous one but specifies zero potential. This boundary condition is also applicable at symmetry boundaries where the potential is known to be antisymmetric with respect to the boundary.

CURRENT SOURCE

You can apply the current-source boundary condition

$$\mathbf{n} \cdot (\mathbf{J}_1 - \mathbf{J}_2) = J_n$$

to interior boundaries that represent either a current source or sink.

CONTINUITY

The continuity boundary condition

$$\mathbf{n} \cdot (\mathbf{J}_1 - \mathbf{J}_2) = 0$$

specifies that the normal components of the electric current are continuous across the interior boundary.

CONTACT RESISTANCE

You can use the contact resistance boundary condition

$$\mathbf{n} \cdot \mathbf{J}_1 = \frac{\sigma}{d}(V_1 - V_2)$$

$$\mathbf{n} \cdot \mathbf{J}_2 = \frac{\sigma}{d}(V_2 - V_1)$$

to model a thin layer of a resistive material. The layer has the thickness d and the conductivity σ. This boundary condition is only available at the border between the parts in an assembly.

AXIAL SYMMETRY

In axisymmetric models, use the boundary condition for axial symmetry at the symmetry axis $r = 0$.

Line Sources

In 3D you can specify line sources along the edges of a geometry.

LINE CURRENT SOURCE

You can apply a line current source Q_{jl} to edges. This source represents electric current per unit length.

Point Sources

It is possible to add point sources to both 2D and 3D models.

POINT CURRENT SOURCE

You can apply a point current source Q_{j0} to points. This source represents an electric current flowing out of the point.

Application Mode Variables

The following table shows the fundamental fields, all derivable from the electric potential, that are available for postprocessing and for use in equations and boundary conditions.

NAME	TYPE	DESCRIPTION	EXPRESSION
`V`	S	electric potential	V
`sigma`	S	electric conductivity	σ
`sigma`*`ij`*	S	electric conductivity, x_ix_j component	σ_{ij}
`Qj`	S	current source	Q_j
`Je`*`i`*	S	external current density, x_i component	J_i^e
`normJe`	S	external current density, norm	$\sqrt{\mathbf{J}^e \cdot \mathbf{J}^e}$
`Ji`*`i`*	S	potential current density, x_i component	$\sigma_{ij}E_j$
`normJi`	S	potential current density, norm	$\sqrt{\mathbf{J}^i \cdot \mathbf{J}^i}$
`J`*`i`*	S	total current density, x_i component	$J_i^e + J_i^i$
`normJ`	S	total current density, norm	$\sqrt{\mathbf{J} \cdot \mathbf{J}}$

NAME	TYPE	DESCRIPTION	EXPRESSION
`Ei`	S	electric field, x_i component	$-\frac{\partial V}{\partial x_i}$
`normE`	S	electric field, norm	$\sqrt{\mathbf{E}\cdot\mathbf{E}}$
`Q`	S	resistive heating	$\mathbf{J}\cdot\mathbf{E}$
`nJ`	B	current density outflow	$\mathbf{n}\cdot\mathbf{J}$
`Qjl`	E	line current source	Q_{jl}
`Qj0`	P	point current source	Q_{j0}

Note: To form the complete application mode variable names, add a suffix consisting of an underscore and the application mode name (default: dc), for example, `normE_dc`. (This does not apply to the dependent variable for the potential.)

The Electrostatics Application Mode

Applications involving *electrostatics* include high-voltage apparatus, electronic devices, and capacitors. The term "statics" means that the time rate of change of the electric field is slow, and that wavelengths are very large compared to the size of the domain of interest.

The *Electrostatics* application mode is available for 3D, 2D in-plane, and 2D axisymmetric models.

PDE Formulation

COMSOL Multiphysics carries out the modeling of static electric fields using the electric potential V. By combining the definition of potential with Gauss' law and the equation of continuity, it is possible to derive the classic Poisson's equation.

Specifically, under static conditions the electric potential, V, is defined by the relationship

$$\mathbf{E} = -\nabla V$$

Combining this equation with the constitutive relationship $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ between $\mathbf{D}$ and $\mathbf{E}$, it is possible to represent Gauss' law as Poisson's equation

$$-\nabla \cdot (\varepsilon_0 \nabla V - \mathbf{P}) = \rho$$

The In-plane Electrostatics application mode assumes a symmetry where the electric potential varies only in the x and y directions and is constant in the z direction. This implies that the electric field, $\mathbf{E}$, is tangential to the xy-plane. Given this symmetry, you solve the same equation as in the 3D case.

The Axisymmetric Electrostatics application mode considers the situation where the fields and geometry are axially symmetric. In this case the electric potential is constant in the φ direction, which implies that the electric field is tangential to the rz-plane.

Writing the previous equation for ρ in cylindrical coordinates and multiplying it by r to avoid singularities at $r = 0$, the equation becomes

$$-\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial z}\end{bmatrix}^T \cdot \left(r\varepsilon_0 \begin{bmatrix}\frac{\partial V}{\partial r}\\ \frac{\partial V}{\partial z}\end{bmatrix} - r\mathbf{P} \right) = r\rho$$

Application Scalar Variables

There is one application-specific scalar variable in this mode:

PROPERTY	NAME	DEFAULT	DESCRIPTION
ε_0	`epsilon0`	$8.854187817 \cdot 10^{-12}$ F/m	Permittivity of vacuum

Boundary Conditions

The relevant interface condition at interfaces between different media for this mode is

$$\mathbf{n}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$$

In the absence of surface charges, this condition is fulfilled by the natural boundary condition

$$\mathbf{n} \cdot [(\varepsilon_0 \nabla V - \mathbf{P})_1 - (\varepsilon_0 \nabla V - \mathbf{P})_2] = -\mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = 0$$

ELECTRIC DISPLACEMENT

The electric-displacement boundary condition

$$\mathbf{n} \cdot \mathbf{D} = \mathbf{n} \cdot \mathbf{D}_0$$

specifies the normal component of the electric displacement at a boundary.

SURFACE CHARGE

The surface-charge boundary condition

$$-\mathbf{n} \cdot \mathbf{D} = \rho_s, \qquad \mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$$

specifies the surface charge density at an outer boundary or at an interior boundary between two nonconducting media.

ZERO CHARGE/SYMMETRY

The zero charge/symmetry boundary condition

$$\mathbf{n} \cdot \mathbf{D} = 0$$

specifies that the normal component of the electric displacement equals zero.

This boundary condition is also applicable at symmetry boundaries where the potential is known to be symmetric with respect to the boundary.

ELECTRIC POTENTIAL

The electric-potential boundary condition

$$V = V_0$$

specifies the voltage at a boundary. Because you are solving for the potential in this application mode, you generally define the value of the potential at some boundary in the geometry.

GROUND

The ground boundary condition

$$V = 0$$

is a special case of the previous one but specifying zero potential. This boundary condition is also applicable at symmetry boundaries where the potential is known to be antisymmetric with respect to the boundary.

CONTINUITY

The continuity boundary condition

$$\mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = 0$$

specifies that the normal component of the electric displacement is continuous across the boundary.

THIN LOW PERMITTIVITY GAP

You can use the thin low permittivity gap condition

$$\mathbf{n} \cdot \mathbf{D}_1 = \frac{\varepsilon}{d}(V_1 - V_2)$$

$$\mathbf{n} \cdot \mathbf{D}_2 = \frac{\varepsilon}{d}(V_2 - V_1)$$

to model a thin gap of a material with a small permittivity compared to the adjacent domains. The layer has the thickness d and the relative permittivity ε_r. This boundary condition is only available at the border between the parts in an assembly.

AXIAL SYMMETRY

In axisymmetric models, use the boundary condition for axial symmetry on the symmetry axis $r = 0$.

Line Sources

In 3D models you can specify line sources along the edges of a geometry.

LINE CHARGE

You can apply a line charge Q_l along edges. Provide the source as the electric charge per unit length.

Point Sources

It is possible to use point sources in 2D and 3D models.

POINT CHARGE

You can apply a point charge Q_0 at points in a model.

Application Mode Variables

The fundamental fields, which are derivable from the electric potential, are available for postprocessing and for use in equations and boundary conditions. The expressions for some variables in the following table vary depending on the constitutive relationship as indicated in the Const. Rel. column; the abbreviations indicate the constitutive relationship as given in the second table.

NAME	TYPE	CONST. REL.	DESCRIPTION	EXPRESSION
`V`	S		electric potential	V
`epsilon0`	S		permittivity of a vacuum	ε_0
`epsilonr`	S	epsr, Dr	relative permittivity	ε_r
`epsilonr`	S	P	relative permittivity	1
`epsilonr`*`ij`*	S	epsr, Dr	relative permittivity, x_ix_j component	$\varepsilon_{\mathrm{r}ij}$
`epsilonr`*`ij`*	S	P	relative permittivity, x_ix_j component	1

NAME	TYPE	CONST. REL.	DESCRIPTION	EXPRESSION
`epsilon`	S		permittivity	$\varepsilon_0\varepsilon_r$
`epsilon`*ij*	S		permittivity, x_ix_j component	$\varepsilon_0\varepsilon_{rij}$
`P`*i*	S	P	electric polarization, x_i component	P_i
`P`*i*	S	epsr, Dr	electric polarization, x_i component	$D_i - \varepsilon_0 E_i$
`normP`	S		electric polarization, norm	$\sqrt{\mathbf{P}\cdot\mathbf{P}}$
`Dr`*i*	S	epsr	remanent displacement, x_i component	0
`Dr`*i*	S	P	remanent displacement, x_i component	P_i
`Dr`*i*	S	Dr	remanent displacement, x_i component	D_{ri}
`normDr`	S		remanent displacement, norm	$\sqrt{\mathbf{D}_r\cdot\mathbf{D}_r}$
`rho`	S		space-charge density	ρ
`E`*i*	S		electric field, x_i component	$-\frac{\partial V}{\partial x_i}$
`normE`	S		electric field, norm	$\sqrt{\mathbf{E}\cdot\mathbf{E}}$
`D`*i*	S	epsr	electric displacement, x_i component	$\varepsilon_{ij}E_j$
`D`*i*	S	P	electric displacement, x_i component	$\varepsilon_0 E_i + P_i$
`D`*i*	S	Dr	electric displacement, x_i component	$\varepsilon_0\varepsilon_{rij}E_j + D_{ri}$
`normD`	S		electric displacement, norm	$\sqrt{\mathbf{D}\cdot\mathbf{D}}$
`We`	S		electric-energy density	$\frac{\mathbf{E}\cdot\mathbf{D}}{2}$
`nD`	B		surface-charge density	$-\mathbf{n}\cdot\mathbf{D}$
`Ql`	E		line-charge density	Q_l
`Q0`	P		charge	Q_0

Note: To form the complete application mode variable names, add a suffix consisting of an underscore and the application mode name (default: es), for example, `normE_es`. (This does not apply to the dependent variable for the potential.)

TABLE 5-1: ABBREVIATIONS FOR THE CONSTITUTIVE RELATIONSHIPS

ABBREVIATION	CONSTITUTIVE RELATIONSHIP
`epsr`	$\mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E}$
`P`	$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$
`Dr`	$\mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E} + \mathbf{D}_r$

Magnetostatics Application Mode

It is often possible to model magnets, electric motors, and transformers with *magnetostatics*. The term "statics" implies that the time rate of change of the magnetic field is slow.

The *Magnetostatics* application mode is available for 2D in-plane and 2D axisymmetric models.

PDE Formulation

To derive the equation system this mode solves, start with Ampère's law for static cases,

$$\nabla \times \mathbf{H} = \mathbf{J}$$

The current is

$$\mathbf{J} = \sigma \mathbf{v} \times \mathbf{B} + \mathbf{J}^e$$

where $\mathbf{J}^e$ is an externally generated current density, and $\mathbf{v}$ is the velocity of the conductor.

Using the definitions of magnetic potential,

$$\mathbf{B} = \nabla \times \mathbf{A}$$

and the constitutive relationship, $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$, you can rewrite Ampère's law as

$$\nabla \times (\mu_0^{-1} \nabla \times \mathbf{A} - \mathbf{M}) - \sigma \mathbf{v} \times (\nabla \times \mathbf{A}) = \mathbf{J}^e$$

In the 2D case there are no variations in the z direction, and the current is parallel to the z-axis. Therefore you can add a term $-\sigma \Delta V / L$ to the definition of the current where ΔV is the potential difference over the distance L. This leads to

$$\mathbf{J} = \sigma \mathbf{v} \times \mathbf{B} + \mathbf{J}^e - \sigma \frac{\Delta V}{L}$$

In 2D, that equation simplifies

$$-\nabla \cdot \left(\mu_0^{-1} \nabla A_z - \begin{bmatrix} -M_y \\ M_x \end{bmatrix} \right) + \sigma \mathbf{v} \cdot \nabla A_z = \sigma \frac{\Delta V}{L} + J_z^e$$

The axisymmetric case uses another form of the contribution of the current coming from a potential difference $-\sigma(V_{\text{loop}}/(2\pi r))$ because current is present only in the azimuthal direction. The above equation then becomes, in cylindrical coordinates,

$$-\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial z}\end{bmatrix}^T\left(r\mu_0^{-1}\begin{bmatrix}\frac{\partial u}{\partial r}\\ \frac{\partial u}{\partial z}\end{bmatrix}+\mu_0^{-1}\begin{bmatrix}2\\0\end{bmatrix}u-\begin{bmatrix}M_z\\-M_r\end{bmatrix}\right)+r\sigma\left(\mathbf{v}\cdot\begin{bmatrix}\frac{\partial u}{\partial r}\\ \frac{\partial u}{\partial z}\end{bmatrix}\right)+2\sigma v_r u = \sigma\frac{V_{\text{loop}}}{2\pi r}+J_\varphi^e$$

The dependent variable u is the nonzero component of the magnetic potential divided by the radial coordinate r, that is,

$$u = \frac{A_\varphi}{r}$$

The application mode performs this transformation to avoid singularities at the symmetry axis.

Application Mode Property

The application mode property appears in the following table:

PROPERTY	VALUES	DESCRIPTION
Analysis type	Static | Time-harmonic	Specifies which type of analysis to perform

Application Scalar Variable

The application scalar variable in this application mode is the permeability of vacuum.

PROPERTY	NAME	VALUE	DESCRIPTION
μ_0	`mu0`	$4\pi\cdot 10^{-7}$ H/m	Permeability of vacuum

Boundary and Interface Conditions

The relevant interface conditions are

$$\mathbf{n}_2\times(\mathbf{H}_1-\mathbf{H}_2) = \mathbf{J}_s$$

The natural boundary condition fulfills this equation if the surface current vanishes. You can then transform the Neumann condition of the previous PDE to

$$\mathbf{n}\cdot\left(\mu_0^{-1}\nabla A_z - \begin{bmatrix}-M_y\\ M_x\end{bmatrix}\right) = -\mathbf{n}\times(\mu_0^{-1}\nabla\times\mathbf{A}-\mathbf{M}) = -\mathbf{n}\times\mathbf{H} = \mathbf{0}$$

MAGNETIC FIELD

The magnetic-field boundary condition

$$\mathbf{n}\times\mathbf{H} = \mathbf{n}\times\mathbf{H}_0$$

specifies the tangential component of the magnetic field at the boundary.

SURFACE CURRENT

The surface current boundary condition

$$-\mathbf{n}\times\mathbf{H} = J_{sz}\mathbf{e}_z \qquad \mathbf{n}\times(\mathbf{H}_1-\mathbf{H}_2) = J_{sz}\mathbf{e}_z$$

specifies a surface current flowing in the z direction.

ELECTRIC INSULATION

The electric-insulation boundary condition

$$\mathbf{n}\times\mathbf{H} = \mathbf{0}$$

sets the magnetic field to zero. The term *electric insulation* arises from the fact that this boundary condition makes the normal component of the electric current equal to zero.

MAGNETIC POTENTIAL

The magnetic-potential boundary condition

$$A_z = A_{0z}$$

specifies the magnetic potential.

MAGNETIC INSULATION

The magnetic-insulation boundary condition

$$A_z = 0$$

sets the magnetic potential to zero at the boundary. This boundary condition is also applicable at symmetry boundaries where the magnetic field is known to be tangential to the boundary.

The term *magnetic insulation* comes from the fact that this boundary condition makes the normal component of the magnetic field equal to zero. Thus the boundary is not truly insulating.

CONTINUITY

The continuity boundary condition

$$\mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{0}$$

is the natural boundary condition implying continuity of the tangential component of the magnetic field.

Point Sources

You can specify that points in the geometry carry a current I_0 flowing in the z direction.

Application Mode Variables

For time-harmonic analysis, all fundamental field quantities are available for postprocessing and for use in equations and boundary conditions. The same holds true for the magnetic potential and the time average of energy and heating expressions.

At the edges of a modeled structure you have access to the magnetic potential, electric field, and tangential magnetic field for postprocessing. The energy flow in the normal direction is also available.

The expressions for some variables in the following table vary depending on the constitutive relationship as indicated in the Const. Rel. column; the abbreviations indicate the constitutive relationship as given in the second table.

NAME	TYPE	CONST. REL.	DESCRIPTION	EXPRESSION
`Az`	S		magnetic potential, z component	A_z
`mu0`	S		permeability of a vacuum	μ_0
`mur`	S	mur, Br	relative permeability	μ_r

NAME	TYPE	CONST. REL.	DESCRIPTION	EXPRESSION
`mur`	S	M	relative permeability	1
`mur`*`ij`*	S	mur, Br	relative permeability, $x_i x_j$ component	$\mu_{\mathrm{r}ij}$
`mur`*`ij`*	S	M	relative permeability, $x_i x_j$ component	1
`mu`	S		permeability	$\mu_0\mu_\mathrm{r}$
`mu`*`ij`*	S		permeability, $x_i x_j$ component	$\mu_0\mu_{\mathrm{r}ij}$
`sigma`	S		electric conductivity	σ
`deltaV`	S		potential difference	ΔV
`L`	S		length	L
`M`*`i`*	S	M	magnetization, x_i component	M_i
`M`*`i`*	S	mur, Br	magnetization, x_i component	$B_i/\mu_0 - H_i$
`normM`	S		magnetization, norm	$\sqrt{\mathbf{M}\cdot\mathbf{M}^*}$
`Br`*`i`*	S	mur	remanent flux density, x_i component	0
`Br`*`i`*	S	M	remanent flux density, x_i component	$\mu_0 M_i$
`Br`*`i`*	S	Br	remanent flux density, x_i component	$B_{\mathrm{r}i}$
`normBr`	S		remanent flux density, norm	$\sqrt{\mathbf{B}_\mathrm{r}\cdot\mathbf{B}_\mathrm{r}^*}$
`Jez`	S		external current density, z component	J_z^e

NAME	TYPE	CONST. REL.	DESCRIPTION	EXPRESSION
`vi`	S		velocity, x_i component	v_i
`normv`	S		velocity, norm	$\sqrt{\mathbf{v}\cdot\mathbf{v}}$
`Bx`	S		magnetic flux density, x component	$\frac{\partial A_z}{\partial y}$
`By`	S		magnetic flux density, y component	$-\frac{\partial A_z}{\partial x}$
`normB`	S		magnetic flux density, norm	$\sqrt{\mathbf{B}\cdot\mathbf{B}^*}$
`Hi`	S	mur	magnetic field, x_i component	$\mu_{rij}^{-1}B_j/\mu_0$
`Hi`	S	M	magnetic field, x_i component	B_i/μ_0-M_i
`Hi`	S	Br	magnetic field, x_i component	$\mu_{rij}^{-1}(B_j-B_{rj})/\mu_0$
`normH`	S		magnetic field, norm	$\sqrt{\mathbf{H}\cdot\mathbf{H}^*}$
`Jpz`	S		potential current, z component	$\sigma\Delta V/L$
`Jvz`	S		velocity current density, z component	$\sigma(v_xB_y-v_yB_x)$
`Jz`	S		total current density, z component	$J_z^e+J_z^v+J_z^p$
`normJ`	S		total current density, norm	$\lvert J_z\rvert$
`Wm`	S		magnetic energy density	$\frac{\mathbf{H}\cdot\mathbf{B}}{2}$
`Q`	S		resistive heating	$J_z\left(v_xB_y-v_yB_x+\frac{\Delta V}{L}+\sigma^{-1}J_z^e\right)$
`Jsz`	B		surface current density, z component	$n_yH_x-n_xH_y$

NAME	TYPE	CONST. REL.	DESCRIPTION	EXPRESSION
unT*i*	B		Maxwell surface stress tensor, x_i component, up side of boundary	$-\frac{1}{2}(\mathbf{H}_{\mathrm{up}}\cdot\mathbf{B}_{\mathrm{up}})n_{i\mathrm{down}} + (\mathbf{n}_{\mathrm{down}}\cdot\mathbf{H}_{\mathrm{up}})B_{i\mathrm{up}}$
dnT*i*	B		Maxwell surface stress tensor, x_i component, down side of boundary	$-\frac{1}{2}(\mathbf{H}_{\mathrm{down}}\cdot\mathbf{B}_{\mathrm{down}})n_{i\mathrm{up}} + (\mathbf{n}_{\mathrm{up}}\cdot\mathbf{H}_{\mathrm{down}})B_{i\mathrm{down}}$

NAME	TYPE	CONST. REL.	DESCRIPTION	EXPRESSION
Aphidr	S		magnetic potential divided by r	u
Aphi	S		magnetic potential, φ component	ru
Aphir	S		magnetic potential, r derivative of φ component	$r\frac{\partial u}{\partial r} + u$
Aphiz	S		magnetic potential, z derivative of φ component	$r\frac{\partial u}{\partial z}$
mu0	S		permeability of a vacuum	μ_0
mur	S	mur, Br	relative permeability	μ_{r}
mur	S	M	relative permeability	1
mur*ij*	S	mur, Br	relative permeability, x_ix_j component	$\mu_{\mathrm{r}ij}$
mur*ij*	S	M	relative permeability, x_ix_j component	1
mu	S		permeability	$\mu_0\mu_{\mathrm{r}}$

NAME	TYPE	CONST. REL.	DESCRIPTION	EXPRESSION
mu*ij*	S		permeability, x_ix_j component	$\mu_0\mu_{rij}$
sigma	S		electric conductivity	σ
Vloop	S		loop potential	V_{loop}
M*i*	S	M	magnetization, x_i component	M_i
M*i*	S	mur, Br	magnetization, x_i component	$B_i/\mu_0 - H_i$
normM	S		magnetization, norm	$\sqrt{\mathbf{M}\cdot\mathbf{M}^*}$
Br*i*	S	mur	remanent flux density, x_i component	0
Br*i*	S	M	remanent flux density, x_i component	$\mu_0 M_i$
Br*i*	S	Br	remanent flux density, x_i component	B_{ri}
normBr	S		remanent flux density, norm	$\sqrt{\mathbf{B}_r\cdot\mathbf{B}_r^*}$
Jephi	S		external current density, φ component	J_φ^e
v*i*	S		velocity, x_i component	v_i
normv	S		velocity, norm	$\sqrt{\mathbf{v}\cdot\mathbf{v}}$
normE	S		electric field, norm	$\lvert E_\varphi\rvert$
Br	S		magnetic flux density, r component	$-\frac{\partial A_\varphi}{\partial z}$
Bz	S		magnetic flux density, z component	$u + \frac{\partial A_\varphi}{\partial r}$

NAME	TYPE	CONST. REL.	DESCRIPTION	EXPRESSION
normB	S		magnetic flux density, norm	$\sqrt{\mathbf{B}\cdot\mathbf{B}^*}$
H*i*	S	mur	magnetic field, x_i component	$\mu_{rij}^{-1}B_j/\mu_0$
H*i*	S	M	magnetic field, x_i component	$B_i/\mu_0 - H_i$
H*i*	S	Br	magnetic field, x_i component	$\mu_{rij}^{-1}(B_j - B_{rj})/\mu_0$
normH	S		magnetic field, norm	$\sqrt{\mathbf{H}\cdot\mathbf{H}^*}$
Jpphi	S		loop current, φ component	$\sigma V_{\mathrm{loop}}/2\pi r$
Jiphi	S		induced current density, φ component	σE_{φ}
Jvphi	S		velocity current density, φ component	$\sigma(v_z B_r - v_r B_z)$
Jphi	S		total current density, φ component	$J_{\varphi}^{e} + J_{\varphi}^{v} + J_{\varphi}^{p}$
normJ	S		total current density, norm	$\|J_{\varphi}\|$
Wm	S		magnetic energy density	$\frac{\mathbf{H}\cdot\mathbf{B}}{2}$
Q	S		resistive heating	$J_{\varphi}\left(v_r B_z - v_z B_r + \frac{V_{\mathrm{loop}}}{2\pi r} + \sigma^{-1}J_{\varphi}^{e}\right)$
Jsphi	B		surface current density, φ component	$n_r H_z - n_z H_r$

NAME	TYPE	CONST. REL.	DESCRIPTION	EXPRESSION
`unT`*`i`*	B		Maxwell surface stress tensor, x_i component, up side of boundary	$-\frac{1}{2}(\mathbf{H}_{\mathrm{up}} \cdot \mathbf{B}_{\mathrm{up}})n_{i\mathrm{down}} + (\mathbf{n}_{\mathrm{down}} \cdot \mathbf{H}_{\mathrm{up}})B_{i\mathrm{up}}$
`dnT`*`i`*	B		Maxwell surface stress tensor, x_i component, down side of boundary	$-\frac{1}{2}(\mathbf{H}_{\mathrm{down}} \cdot \mathbf{B}_{\mathrm{down}})n_{i\mathrm{up}} + (\mathbf{n}_{\mathrm{up}} \cdot \mathbf{H}_{\mathrm{down}})B_{i\mathrm{down}}$

Note: To form the complete application mode variable names, add a suffix consisting of an underscore and the application mode name (default: `qa`), for example, `normH_qa`. (This does not apply to the dependent variables for the magnetic potential's z component.)

TABLE 5-2: ABBREVIATIONS FOR THE CONSTITUTIVE RELATIONSHIPS

ABBREVIATION	CONSTITUTIVE RELATIONSHIP
`mur`	$\mathbf{B} = \mu_0\mu_r\mathbf{H}$
`M`	$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$
`Br`	$\mathbf{B} = \mu_0\mu_r\mathbf{H} + \mathbf{B}_r$

AC Power Electromagnetics

AC power electromagnetics problems are common when studying motors, transformers, and conductors carrying alternating currents.

You can use the *AC Power Electromagnetics* application mode for 2D in-plane and 2D axisymmetric models.

PDE Formulation

To derive the equation system this mode solves, start with Ampère's law,

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = \sigma \mathbf{E} + \sigma \mathbf{v} \times \mathbf{B} + \mathbf{J}^e + \frac{\partial \mathbf{D}}{\partial t} .$$

Now assume time-harmonic fields and use the definitions of the potentials,

$$\mathbf{B} = \nabla \times \mathbf{A}$$
$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

and combine them with the constitutive relationships $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ and $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$ to rewrite Ampère's law as

$$(j\omega\sigma - \omega^2\varepsilon_0)\mathbf{A} + \nabla \times (\mu_0^{-1} \nabla \times \mathbf{A} - \mathbf{M}) - \sigma \mathbf{v} \times (\nabla \times \mathbf{A}) + (\sigma + j\omega\varepsilon_0)\nabla V = \mathbf{J}^e + j\omega \mathbf{P}$$

In the 2D in-plane case there are no variations in the z direction, and the electric field is parallel to the z-axis. Therefore you can write ∇V as $-\Delta V/L$ where ΔV is the potential difference over the distance L.

Now simplify these equations to

$$-\nabla \cdot \left(\mu_0^{-1} \nabla A_z - \begin{bmatrix} -M_y \\ M_x \end{bmatrix} \right) + \sigma \mathbf{v} \cdot \nabla A_z + (j\omega\sigma - \omega^2\varepsilon_0) A_z = \sigma \frac{\Delta V}{L} + J_z^e + j\omega P_z .$$

The axisymmetric case uses another form of the gradient of the electric potential, $\nabla V = -V_{\mathrm{loop}}/(2\pi r)$, because the electric field is present only in the azimuthal direction. The above equation, in cylindrical coordinates, becomes

$$-\left(\begin{bmatrix}\frac{\partial}{\partial r} & \frac{\partial}{\partial z}\end{bmatrix}\cdot\left(r\mu_0^{-1}\begin{bmatrix}\frac{\partial u}{\partial r}\\ \frac{\partial u}{\partial z}\end{bmatrix}+\mu_0^{-1}\begin{bmatrix}2\\0\end{bmatrix}u-\begin{bmatrix}M_z\\-M_r\end{bmatrix}\right)\right)+r\sigma\left(\mathbf{v}\cdot\begin{bmatrix}\frac{\partial u}{\partial r}\\ \frac{\partial u}{\partial z}\end{bmatrix}\right)+r(\sigma j\omega-\omega^2\varepsilon_0)u+2\sigma v_r u$$

$$=\sigma\frac{V_{\mathrm{loop}}}{2\pi r}+J_\varphi^e+j\omega P_\varphi$$

The dependent variable u is the nonzero component of the magnetic potential divided by the radial coordinate r, so that

$$u = \frac{A_\varphi}{r}$$

The application mode performs this transformation to avoid singularities on the symmetry axis.

Application Scalar Variables

The application scalar variables in this application mode are:

PROPERTY	NAME	DEFAULT	DESCRIPTION
μ_0	`mu0`	$4\pi\cdot 10^{-7}$ H/m	Permeability of vacuum
ε_0	`epsilon0`	$8.854187817\cdot 10^{-12}$ F/m	Permittivity of vacuum
ν	`nu`	50 Hz	Frequency

The frequency occurs as a variable in time-harmonic problems.

Boundary and Interface Conditions

The relevant interface condition is

$$\mathbf{n}_2\times(\mathbf{H}_1-\mathbf{H}_2)=\mathbf{J}_s$$

The natural boundary condition fulfills this equation if the surface current vanishes. You can transform the Neumann condition of this PDE into

$$\mathbf{n}\cdot\left[\left(\mu_0^{-1}\nabla A_z-\begin{bmatrix}-M_y\\M_x\end{bmatrix}\right)_1-\left(\mu_0^{-1}\nabla A_z-\begin{bmatrix}-M_y\\M_x\end{bmatrix}\right)_2\right]$$

$$=-\mathbf{n}\times[(\mu_0^{-1}\nabla\times\mathbf{A}-\mathbf{M})_1-(\mu_0^{-1}\nabla\times\mathbf{A}-\mathbf{M})_2]$$

$$=-\mathbf{n}\times(\mathbf{H}_1-\mathbf{H}_2)=\mathbf{0}$$

MAGNETIC FIELD

The magnetic-field boundary condition

$$\mathbf{n}\times\mathbf{H}=\mathbf{n}\times\mathbf{H}_0$$

specifies the tangential component of the magnetic field at the boundary.

SURFACE CURRENT

The surface-current boundary condition

$$-\mathbf{n}\times\mathbf{H}=J_{sz}\mathbf{e}_z\qquad\mathbf{n}\times(\mathbf{H}_1-\mathbf{H}_2)=J_{sz}\mathbf{e}_z$$

lets you specify a surface current flowing in the z direction.

ELECTRIC INSULATION

The electric-insulation boundary condition

$$\mathbf{n}\times\mathbf{H}=\mathbf{0}$$

sets the magnetic field to zero. The term *electric insulation* comes from the fact that this boundary condition makes the normal component of the electric current equal to zero.

MAGNETIC POTENTIAL

The magnetic-potential boundary condition

$$A_z=A_{0z}$$

specifies the magnetic potential.

MAGNETIC INSULATION

The magnetic insulation boundary condition

$$A_z=0$$

sets the magnetic potential to zero at the boundary. This boundary condition is also applicable at symmetry boundaries where the magnetic field is known to be tangential to the boundary.

The term *magnetic insulation* comes from the fact that this boundary condition makes the normal component of the magnetic field zero. Thus the boundary is not truly insulating.

CONTINUITY

The continuity boundary condition

$$\mathbf{n}\times(\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{0}$$

is the natural boundary condition implying continuity of the tangential component of the magnetic field.

Point Sources

You can specify that points in the geometry carry a current I_0 flowing in the z direction.

Application Mode Variables

For time-harmonic analysis, all the fundamental field quantities are available for postprocessing and for use in equations and boundary conditions. The same holds true for the magnetic potential and the time average of energy and heating expressions.

At the edges of the modeled structure you can also use the magnetic potential, electric field and tangential magnetic field for postprocessing. The energy flow in the normal direction is also available.

The expressions for some variables in the following table vary depending on the constitutive relationship as indicated in the Const. Rel. column; the abbreviations indicate the constitutive relationship as given in the second table.'

NAME	TYPE	CONST. REL.	DESCRIPTION	EXPRESSION
Az	S		magnetic potential, z component	A_z
nu	S		frequency	ν
omega	S		angular frequency	$2\pi\nu$

NAME	TYPE	CONST. REL.	DESCRIPTION	EXPRESSION
`epsilon0`	S		permittivity of a vacuum	ε_0
`mu0`	S		permeability of a vacuum	μ_0
`mur`	S	mur, Br	relative permeability	μ_r
`mur`	S	M	relative permeability	1
`mur`*`ij`*	S	mur, Br	relative permeability, x_ix_j component	$\mu_{\mathrm{r}ij}$
`mur`*`ij`*	S	M	relative permeability, x_ix_j component	1
`mu`	S		permeability	$\mu_0\mu_\mathrm{r}$
`mu`*`ij`*	S		permeability, x_ix_j component	$\mu_0\mu_{\mathrm{r}ij}$
`epsilonr`	S	epsr, Dr	relative permittivity	ε_r
`epsilonr`	S	P	relative permittivity	1
`epsilon`	S		permittivity	$\varepsilon_0\varepsilon_\mathrm{r}$
`sigma`	S		electric conductivity	σ
`deltaV`	S		potential difference	ΔV
`L`	S		length	L
`Pz`	S	P	electric polarization, z component	$P_ze^{j\mathrm{phase}}$
`Pz`	S	epsr, Dr	electric polarization, z component	$D_z-\varepsilon_0E_z$
`Drz`	S	epsr	remanent displacement, z component	0

NAME	TYPE	CONST. REL.	DESCRIPTION	EXPRESSION
`Drz`	S	P	remanent displacement, z component	P_z
`Drz`	S	Dr	remanent displacement, z component	$D_{\mathrm{r}z}e^{j\mathrm{phase}}$
`M`*`i`*	S	M	magnetization, x_i component	$M_ie^{j\mathrm{phase}}$
`M`*`i`*	S	mur, Br	magnetization, x_i component	$B_i/\mu_0 - H_i$
`normM`	S		magnetization, norm	$\sqrt{\mathbf{M}\cdot\mathbf{M}^*}$
`Br`*`i`*	S	mur	remanent flux density, x_i component	0
`Br`*`i`*	S	M	remanent flux density, x_i component	μ_0M_i
`Br`*`i`*	S	Br	remanent flux density, x_i component	$B_{\mathrm{r}i}e^{j\mathrm{phase}}$
`normBr`	S		remanent flux density, norm	$\sqrt{\mathbf{B}_r\cdot\mathbf{B}_r^*}$
`Jez`	S		external current density, z component	$J_z^ee^{j\mathrm{phase}}$
`v`*`i`*	S		velocity, x_i component	v_i
`normv`	S		velocity, norm	$\sqrt{\mathbf{v}\cdot\mathbf{v}}$
`Ez`	S		electric field, z component	$-j\omega A_z$
`normE`	S		electric field, norm	$\lvert E_z\rvert$
`Dz`	S	epsr	electric displacement, z component	$\varepsilon_0\varepsilon_{\mathrm{r}}E_z$

NAME	TYPE	CONST. REL.	DESCRIPTION	EXPRESSION
`Dz`	S	P	electric displacement, z component	$\varepsilon_0 E_z + P_z$
`Dz`	S	Dr	electric displacement, z component	$\varepsilon_0 \varepsilon_\mathrm{r} E_z + D_{\mathrm{r}z}$
`normD`	S		electric displacement, norm	$\lvert D_z \rvert$
`Bx`	S		magnetic flux density, x component	$\frac{\partial A_z}{\partial y}$
`By`	S		magnetic flux density, y component	$-\frac{\partial A_z}{\partial x}$
`normB`	S		magnetic flux density, norm	$\sqrt{\mathbf{B} \cdot \mathbf{B}^*}$
`H`*i*	S	mur	magnetic field, x_i component	$\mu_{rij}^{-1} B_j / \mu_0$
`H`*i*	S	M	magnetic field, x_i component	$B_i / \mu_0 - M_i$
`H`*i*	S	Br	magnetic field, x_i component	$\mu_{rij}^{-1}(B_j - B_{rj}) / \mu_0$
`normH`	S		magnetic field, norm	$\sqrt{\mathbf{H} \cdot \mathbf{H}^*}$
`Jpz`	S		potential current, z component	$\sigma \Delta V / L$
`Jiz`	S		induced current density, z component	σE_z
`Jdz`	S		displacement current density, z component	$j\omega D_z$
`Jvz`	S		velocity current density, z component	$\sigma(v_x B_y - v_y B_x)$

NAME	TYPE	CONST. REL.	DESCRIPTION	EXPRESSION
`Jz`	S		total current density, z component	$J_z^e + J_z^v + J_z^p + J_z^i + J_z^d$
`normJ`	S		total current density, norm	$\|J_z\|$
`Weav`	S		time average electric energy density	$\frac{1}{4}\mathrm{Re}(E_z D_z^*)$
`Wmav`	S		time average magnetic energy density	$\frac{1}{4}\mathrm{Re}(\mathbf{H} \cdot \mathbf{B}^*)$
`Wav`	S		time average total energy density	$W_e^{\mathrm{av}} + W_m^{\mathrm{av}}$
`Qav`	S		time average resistive heating	$\frac{1}{2}\mathrm{Re}\Big(J_z\Big(E_z^* + v_x B_y^* - v_y B_x^* + \frac{\Delta V^*}{L} + \sigma^{-1} J_z^{e*}\Big)\Big)$
`Poxav`	S		time average power flow, x component	$-\frac{1}{2}\mathrm{Re}(E_z H_y^*)$
`Poyav`	S		time average power flow, y component	$\frac{1}{2}\mathrm{Re}(E_z H_x^*)$
`normPoav`	S		time average power flow, norm	$\sqrt{\mathbf{S}^{\mathrm{av}} \cdot \mathbf{S}^{\mathrm{av}*}}$
`Jsz`	B		surface current density, z component	$n_y H_x - n_x H_y$
`nPoav`	B		time average power outflow	$\mathbf{n} \cdot \mathbf{S}^{\mathrm{av}}$
`unT`*`i`*`av`	B		Maxwell surface stress tensor, x_i component, up side of boundary	$-\frac{1}{4}\mathrm{Re}(\mathbf{H}_{\mathrm{up}} \cdot \mathbf{B}_{\mathrm{up}}^*) n_{i\mathrm{down}} + \frac{1}{2}\mathrm{Re}((\mathbf{n}_{\mathrm{down}} \cdot \mathbf{H}_{\mathrm{up}}) B_{i\mathrm{up}}^*)$
`dnT`*`i`*`av`	B		Maxwell surface stress tensor, x_i component, down side of boundary	$-\frac{1}{4}\mathrm{Re}(\mathbf{H}_{\mathrm{down}} \cdot \mathbf{B}_{\mathrm{down}}^*) n_{i\mathrm{up}} + \frac{1}{2}\mathrm{Re}((\mathbf{n}_{\mathrm{up}} \cdot \mathbf{H}_{\mathrm{down}}) B_{i\mathrm{down}}^*)$

NAME	TYPE	CONST. REL.	DESCRIPTION	EXPRESSION
`Aphidr`	S		magnetic potential divided by r	u
`Aphi`	S		magnetic potential, φ component	ru
`Aphir`	S		magnetic potential, r derivative of φ component	$r\frac{\partial u}{\partial r} + u$
`Aphiz`	S		magnetic potential, z derivative of φ component	$r\frac{\partial u}{\partial z}$
`nu`	S		frequency	ν
`omega`	S		angular frequency	$2\pi\nu$
`epsilon0`	S		permittivity of vacuum	ε_0
`mu0`	S		permeability of vacuum	μ_0
`mur`	S	mur, Br	relative permeability	μ_r
`mur`	S	M	relative permeability	1
`mur`*`ij`*	S	mur, Br	relative permeability, x_ix_j component	μ_{rij}
`mur`*`ij`*	S	M	relative permeability, x_ix_j component	1
`mu`	S		permeability	$\mu_0\mu_r$
`mu`*`ij`*	S		permeability, x_ix_j component	$\mu_0\mu_{rij}$
`epsilonr`	S	epsr, Dr	relative permittivity	ε_r
`epsilonr`	S	P	relative permittivity	1

NAME	TYPE	CONST. REL.	DESCRIPTION	EXPRESSION
`epsilon`	S		permittivity	$\varepsilon_0\varepsilon_r$
`sigma`	S		electric conductivity	σ
`Vloop`	S		loop potential	V_{loop}
`Pphi`	S	P	electric polarization, φ component	$P_\varphi e^{j\mathrm{phase}}$
`Pphi`	S	epsr, Dr	electric polarization, φ component	$D_\varphi - \varepsilon_0 E_\varphi$
`Drphi`	S	epsr	remanent displacement, φ component	0
`Drphi`	S	P	remanent displacement, φ component	P_φ
`Drphi`	S	Dr	remanent displacement, φ component	$D_{\mathrm{r}\varphi} e^{j\mathrm{phase}}$
`M`*`i`*	S	M	magnetization, x_i component	$M_i e^{j\mathrm{phase}}$
`M`*`i`*	S	mur, Br	magnetization, x_i component	$B_i/\mu_0 - H_i$
`normM`	S		magnetization, norm	$\sqrt{\mathbf{M}\cdot\mathbf{M}^*}$
`Br`*`i`*	S	mur	remanent flux density, x_i component	0
`Br`*`i`*	S	M	remanent flux density, x_i component	$\mu_0 M_i$
`Br`*`i`*	S	Br	remanent flux density, x_i component	$B_{\mathrm{r}i} e^{j\mathrm{phase}}$
`normBr`	S		remanent flux density, norm	$\sqrt{\mathbf{B}_r\cdot\mathbf{B}_r^*}$

NAME	TYPE	CONST. REL.	DESCRIPTION	EXPRESSION
`Jephi`	S		external current density, φ component	$J^e_\varphi e^{j\mathrm{phase}}$
`vi`	S		velocity, x_i component	v_i
`normv`	S		velocity, norm	$\sqrt{\mathbf{v}\cdot\mathbf{v}}$
`Ephi`	S		electric field, φ component	$-j\omega A_\varphi$
`normE`	S		electric field, norm	$\lvert E_\varphi\rvert$
`Dphi`	S	epsr	electric displacement, φ component	$\varepsilon_0\varepsilon_\mathrm{r}E_\varphi$
`Dphi`	S	P	electric displacement, φ component	$\varepsilon_0 E_\varphi + P_\varphi$
`Dphi`	S	Dr	electric displacement, φ component	$\varepsilon_0\varepsilon_r E_\varphi + D_{r\varphi}$
`normD`	S		electric displacement, norm	$\lvert D_\varphi\rvert$
`Br`	S		magnetic flux density, r component	$-\frac{\partial A_\varphi}{\partial z}$
`Bz`	S		magnetic flux density, z component	$u + \frac{\partial A_\varphi}{\partial r}$
`normB`	S		magnetic flux density, norm	$\sqrt{\mathbf{B}\cdot\mathbf{B}^*}$
`Hi`	S	mur	magnetic field, x_i component	$\mu^{-1}_{rij}B_j/\mu_0$
`Hi`	S	M	magnetic field, x_i component	$B_i/\mu_0 - M_i$
`Hi`	S	Br	magnetic field, x_i component	$\mu^{-1}_{rij}(B_j - B_{rj})/\mu_0$

NAME	TYPE	CONST. REL.	DESCRIPTION	EXPRESSION
normH	S		magnetic field, norm	$\sqrt{\mathbf{H}\cdot\mathbf{H}^*}$
Jpphi	S		loop current, φ component	$\sigma V_{\mathrm{loop}}/2\pi r$
Jiphi	S		induced current density, φ component	σE_φ
Jdphi	S		displacement current density, φ component	$j\omega D_\varphi$
Jvphi	S		velocity current density, φ component	$\sigma(v_z B_r - v_r B_z)$
Jphi	S		total current density, φ component	$J^e_\varphi + J^v_\varphi + J^p_\varphi + J^i_\varphi + J^d_\varphi$
normJ	S		total current density, norm	$\lvert J_\varphi\rvert$
Weav	S		time-average electric energy density	$\frac{1}{4}\mathrm{Re}(E_\varphi D^*_\varphi)$
Wmav	S		time-average magnetic energy density	$\frac{1}{4}\mathrm{Re}(\mathbf{H}\cdot\mathbf{B}^*)$
Wav	S		time-average total energy density	$W_e^{\mathrm{av}} + W_m^{\mathrm{av}}$
Qav	S		time-average resistive heating	$\frac{1}{2}\mathrm{Re}\left(J_\varphi\left(E^*_\varphi + v_r B^*_z - v_z B^*_r + \frac{V^*_{\mathrm{loop}}}{2\pi r} + \sigma^{-1}J^{e*}_\varphi\right)\right)$
Porav	S		time-average power flow, r component	$-\frac{1}{2}\mathrm{Re}(E_\varphi H^*_z)$
Pozav	S		time-average power flow, z component	$\frac{1}{2}\mathrm{Re}(E_\varphi H^*_z)$
normPoav	S		time-average power flow, norm	$\sqrt{\mathbf{S}^{\mathrm{av}}\cdot\mathbf{S}^{\mathrm{av}*}}$

NAME	TYPE	CONST. REL.	DESCRIPTION	EXPRESSION
`Jsphi`	B		surface current density, φ component	$n_r H_z - n_z H_r$
`nPoav`	B		time-average power outflow	$\mathbf{n} \cdot \mathbf{S}^{\mathrm{av}}$
`unTiav`	B		Maxwell surface stress tensor, x_i component, up side of boundary	$-\frac{1}{4}\mathrm{Re}(\mathbf{H}_{\mathrm{up}} \cdot \mathbf{B}^*_{\mathrm{up}}) n_{i\mathrm{down}} + \frac{1}{2}\mathrm{Re}((\mathbf{n}_{\mathrm{down}} \cdot \mathbf{H}_{\mathrm{up}}) B^*_{i\mathrm{up}})$
`dnTiav`	B		Maxwell surface stress tensor, x_i component, down side of boundary	$-\frac{1}{4}\mathrm{Re}(\mathbf{H}_{\mathrm{down}} \cdot \mathbf{B}^*_{\mathrm{down}}) n_{i\mathrm{up}} + \frac{1}{2}\mathrm{Re}((\mathbf{n}_{\mathrm{up}} \cdot \mathbf{H}_{\mathrm{down}}) B^*_{i\mathrm{down}})$

Note: To form the complete application mode variable names, add a suffix consisting of an underscore and the application mode name (default: qa), for example, `Jz_qa`. (This does not apply to the dependent variable for the magnetic potential.)

TABLE 5-3: ABBREVIATIONS FOR THE CONSTITUTIVE RELATIONSHIPS

ABBREVIATION	CONSTITUTIVE RELATIONSHIP
`mur`	$\mathbf{B} = \mu_0\mu_r\mathbf{H}$
`M`	$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$
`Br`	$\mathbf{B} = \mu_0\mu_r\mathbf{H} + \mathbf{B}_r$

Example: Electric Sensor

This is a model from electric impedance tomography, a method of imaging the interior permittivity distribution of an object by measuring current and voltage at the surface. The technique is used in, for example, medical diagnosis. Because different organs have different properties, you can "see" the organs and their movement from the outside.

The model shows how you can determine the shape and the placement of small objects with different material properties inside a closed box from outside. Applying a potential difference on the boundaries of the box gives rise to a surface charge density that varies depending on the permittivity distribution inside the box. By looking at the surface charge density you can therefore see the shape of the different materials inside the box.

Model Definition

This model solves Gauss' law with $\rho = 0$.

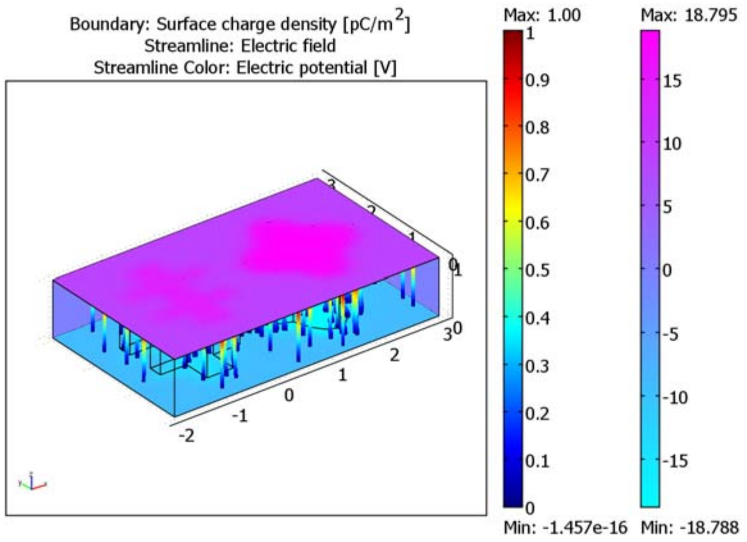
$$-\nabla \cdot (\varepsilon_0 \varepsilon_r \nabla V) = \rho$$

The box contains air with ε_r equal to 1 and the different objects are made of material with different values of the relative permittivity, ε_r: 1, 2, and 3.

To get a voltage difference, set $V = 0$ on the bottom and $V = 1$ on top of the box. On all the other boundaries, use an electric insulation condition: $\mathbf{n} \cdot \mathbf{D} = 0$

Results and Discussion

The surface charge density is higher above material with higher permittivity as expected. You can clearly see the shape of the figures inside the box on the top surface in the following plot.



Inside the geometry the streamlines show how the electric field varies. The gradient of the electric field is lower in media with larger value of the permittivity.

Model Library path: `COMSOL_Multiphysics/Electromagnetics/electric_sensor`

Modeling Using the Graphical User Interface

1 Open COMSOL Multiphysics.
2 In the **Model Navigator**, change the space dimension to **3D** and select **COMSOL Multiphysics>Electromagnetics>Electrostatics** in the list of COMSOL Multiphysics application modes.
3 Click **OK**.

GEOMETRY MODELING

1 Select **Work-Plane Settings** from the **Draw** menu. Set **z** to `0.1` and click **OK**.

2 From the **Options** menu, choose **Axes/Grid Settings**.

3 Enter the following data; when done, click **OK** (to enter the spacings on the Grid page, first clear the **Auto** check box):

AXIS PAGE	
x min	-2
x max	3
y min	0
y max	3
GRID PAGE	
x spacing	0.5
y spacing	0.5

4 Draw a rectangle by clicking on the **Rectangle/Square** button and then clicking at the points (–1, 0.5) and (–0.5, 2.5).

5 Specify two rectangles by choosing **Specify Objects>Rectangle** from the **Draw** menu. Use the following data:

	R2	**R3**
Width	1.5	1.5
Height	0.25	0.25
x	-1.5	-1.5
y	1	1.75

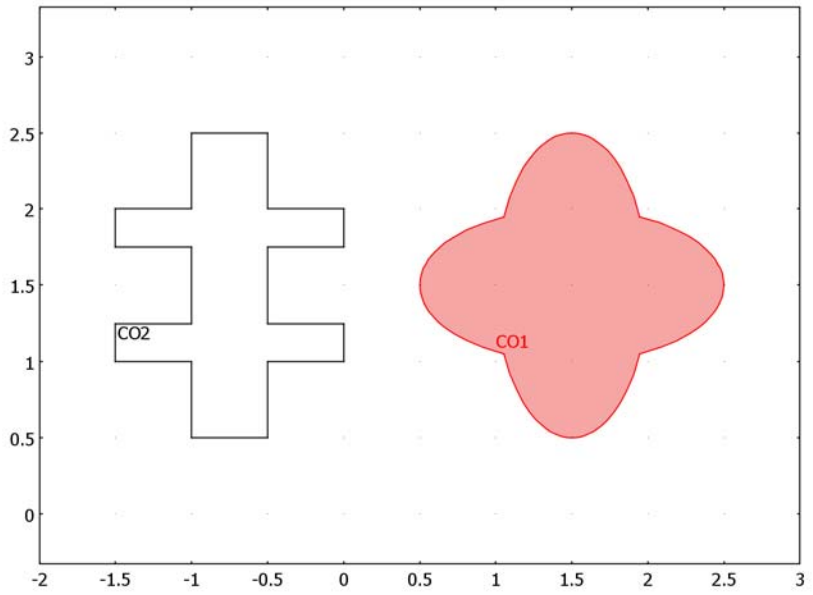
6 Select all objects by pressing Ctrl+A. On the Draw toolbar, click the **Union** button and then the **Delete Interior Boundaries** button.

7 Draw an ellipse by pressing the **Ellipse/Circle (Centered)** button. Click (1.5, 1.5) and (0.5, 1).

8 Draw another ellipse clicking at the points (1.5, 1.5) and (1, 0.5).

9 From the **Draw** menu or the Draw toolbar, open the **Create Composite Object** dialog box and select **E1** and **E2**. Clear the **Keep interior boundaries** check box, and click **OK**.

10 Select **Extrude** from the **Draw** menu, select both objects and enter `0.8` in the **Distance** edit field. Click **OK**.



11 Click the **Block** button. Enter the following dimensions and base point coordinates to create a block:

BLOCK	BLK1	
	Length	Axis base point
x	5	-2
y	3	0
z	1	0

12 Click the **Zoom Extents** button on the Main toolbar.

PHYSICS SETTINGS

Boundary Conditions

1. Open the **Boundary Settings** dialog box from the **Physics** menu.
2. Press Ctrl+A to select all boundaries. Set the boundary condition to **Zero charge/ Symmetry**.
3. Select Boundary 3 and set the boundary condition to **Ground**.

4 Select Boundary 4 and set the boundary condition to **Electric potential** and $\mathbf{V_0}$ to 1.

5 Click **OK**.

Subdomain Settings

In the **Subdomain Settings** dialog box, set the following values of ε_r:

SETTING	SUBDOMAIN 1	SUBDOMAIN 2	SUBDOMAIN 3
ε_r (isotropic)	1	2	3

MESH GENERATION

1 Open the **Free Mesh Parameters** dialog box and select **Fine** from the **Predefined mesh sizes** list on the **General** tab. Click **OK** to close the dialog box.

2 Click the **Initialize Mesh** button on the Main toolbar to initialize the mesh.

COMPUTING THE SOLUTION

Solve the model by clicking the **Solve** button on the Main toolbar.

POSTPROCESSING AND VISUALIZATION

1 Open the **Plot Parameters** dialog box.

2 On the **General** page clear the **Slice** check box and select the **Boundary** and **Streamline** check boxes.

3 Click the **Boundary** tab. In the **Boundary data** area, select **Surface charge density** from the **Predefined quantities** list. In the **Unit** edit field, type `pC/m^2`.

4 In the **Boundary color** area, select **cool** from the **Colormap** list.

5 On the **Streamline** page, click the **Line Color** tab.

6 Click the **Use expression** button. Then click the **Color Expression** button and set the color data to **Electric potential** in the **Predefined quantities** list. Click **OK**.

7 Click the **Start Points** tab. Type `100` in the **Number of start points** edit field.

8 Select **Tube** from the **Line type** list.

9 Click the **Tube Radius** button and set the **Radius data** to **Electric field, norm**. Click **OK**.

10 Click **OK**.

To see the streamlines inside the box you must suppress some of the boundaries:

1 On the **Options** menu, point to **Suppress** and then click **Suppress Boundaries**.

2 In the dialog box select Boundaries 1, 2, and 6–37. Click **OK**.

3 Open the **Plot Parameters** dialog box and click **OK**.

Example: A Permanent Magnet

As an example of a magnetostatics problem, consider how to model a horseshoe-shaped permanent magnet. It consists of a ferromagnetic material, but the two end sections, often painted red and white, are premagnetized in opposite directions. The magnetic field pattern around such a permanent magnet is well known.

The domain consists of four regions:

- Three parts of the permanent magnet—the two premagnetized ends and the curved midsection
- The air surrounding the magnet

The permeability μ in air equals $\mu_0 = 4\pi\cdot10^{-7}$ H/m. Because the magnet is made of a ferromagnetic material, its relative permeability normally depends on the strength of the magnetic field, but in this model it is a constant with a value of 5000. The premagnetization adds a magnetization vector, pointing in the positive x direction at the upper end and in the negative x direction in the lower end. The magnitude of the magnetization is 750 kA/m.

It is reasonable to neglect the field at the boundaries of the computational domain, thus leading to the Dirichlet boundary condition $\mathbf{A} = \mathbf{0}$ on the exterior boundary.

Model Library path: `COMSOL_Multiphysics/Electromagnetics/permanent_magnet`

Modeling Using the Graphical User Interface

MODEL NAVIGATOR

1. Go to the Model Navigator and select **2D** in the **Space dimension** list.
2. Open the **COMSOL Multiphysics>Electromagnetics** folder and then select **Magnetostatics** in the list of application modes.
3. Use the default quadratic Lagrange elements.
4. Click **OK**.

OPTIONS AND SETTINGS

1 Enter the following values in the **Axes/Grid Settings** dialog box. To set the grid spacing, click the **Grid** tab and clear the **Auto** check box.

AXIS		GRID	
x min	`-3`	x spacing	`0.2`
x max	3	Extra x	
y min	`-2`	y spacing	`0.2`
y max	2	Extra y	

2 Enter these names and expressions in the **Constants** dialog box:

NAME	EXPRESSION
`murFe`	5000
`Mpre`	750000

GEOMETRY MODELING

Start by modeling the upper and the lower premagnetized parts of the magnet.

1 Click the **Rectangle/Square** button and draw a rectangle from (0, 0.2) to (0.4, 0.4).

2 Click the **Rectangle/Square** button and draw a rectangle from (0, –0.4) to (0.4, –0.2).

Next draw the rest of the magnet using boundary modeling:

3 Click the **Line** button, and click the points (0.4, 0.2), (0.4, 0.4), and (0.6, 0.4).

4 Click the **2nd Degree Bézier Curve** button, then click at the points (1, 0.4), (1, 0), (1, –0.4), and (0.6, –0.4).

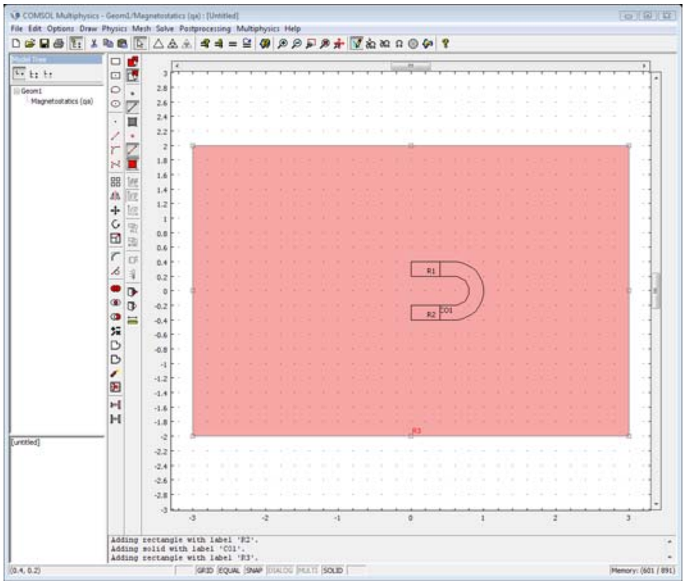
5 Click **Line** and then click at the points (0.4, -0.4), (0.4, –0.2), and (0.6, -0.2).

6 Click **2nd Degree Bézier Curve** and then click at the points (0.8, –0.2), (0.8, 0), (0.8, 0.2), and (0.6, 0.2).

7 Close the region with the right mouse button.

The last step in creating the geometry is to model the surrounding air.

8 Draw a rectangle from (–3, –2) to (3, 2).


COMSOL Multiphysics - Geom1/Magnetostatics (qa) : [Untitled]
File Edit Options Draw Physics Mesh Solve Postprocessing Multiphysics Help
R1
R2
CO1
R3
GRID EQUAL SNAP DIALOG MULTI SOLID
(0.4, 0.2)
Memory: (601 / 891)

```
Adding rectangle with label 'R2'.
Adding solid with label 'CO1'.
Adding rectangle with label 'R3'.
```

PHYSICS SETTINGS

Boundary Conditions

Use the default magnetic insulation boundary condition everywhere.

Subdomain Settings

The relative permeability for the magnet is 5000, and the magnetization vector is (M_{pre}, 0) in the upper premagnetized part and ($-M_{\mathrm{pre}}$, 0) in the lower part.

1 Go to the **Physics** menu and choose **Subdomain Settings**.

2 Enter the PDE coefficients as shown in the following table. For Subdomains 1 and 4 use the default constitutive relationship $\mathbf{B} = \mu_0\mu_r\mathbf{H}$, and for Subdomains 2 and 3 use the constitutive relationship $B = \mu_0\mathbf{H} + \mu_0\mathbf{M}$.

SETTINGS	SUBDOMAIN 1	SUBDOMAIN 2	SUBDOMAIN 3	SUBDOMAIN 4
μ_r	1			`murFe`
M_x		`-Mpre`	`Mpre`	
M_y		`0`	`0`	

MESH GENERATION

To resolve the field at the ends of the magnets, use a smaller mesh size at those boundaries:

1 From the **Mesh** menu, choose **Free Mesh Parameters**.
2 Click the **Boundary** tab.
3 Select Boundaries 4 and 7 (the ends of the magnet).
4 Type `0.01` in the **Maximum element size** edit field.
5 Click **OK**.
6 Click the **Initialize Mesh** button on the Main toolbar to create the mesh.

COMPUTING THE SOLUTION

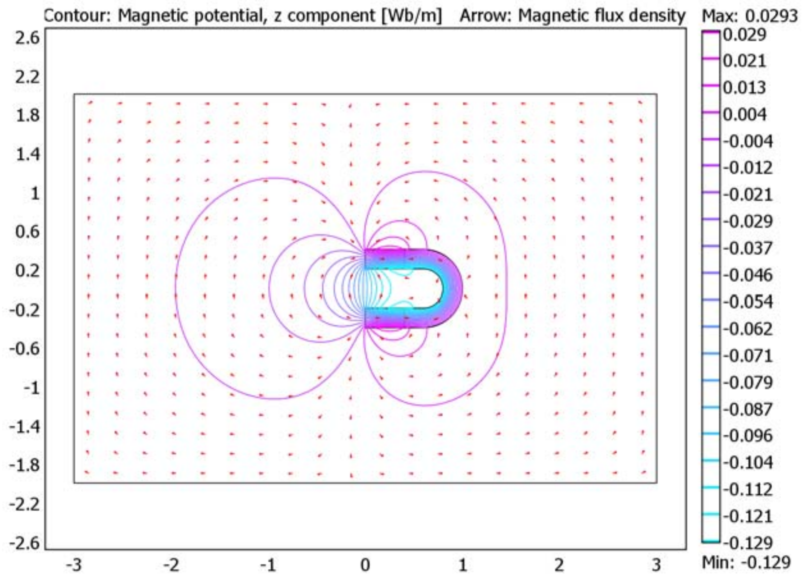
Use the adaptive solver to improve the solution accuracy.

1 Go to the **Solve** menu and choose **Solver Parameters**.
2 In the **Solver Parameters** dialog box, select the **Adaptive mesh refinement** check box underneath the **Solver** list.
3 Click **OK**.
4 Click the **Solve** button.

POSTPROCESSING AND VISUALIZATION

Use the streamline plot to visualize the magnetic field using field lines where the distance between the lines is inversely proportional to the magnetic field strength.

1 Open the **Plot Parameters** dialog box.
2 Select the **Streamline** check box.
3 Click the **Streamline** tab.
4 Select **Magnitude controlled** from the **Streamline plot type** list.
5 Type `50` in the **Density** edit field.
6 Click the **Line Color** tab.
7 Click the **Color** button, and then select a white color from the palette in the **Streamline Color** dialog box. Click **OK**.
8 Click **OK**.



The resulting plot shows the well-known field pattern.

Alternative Modeling Approach

An alternate way to model a permanent magnet is to set surface currents on the premagnetized parts. To do so, change the model from the previous discussion in the following fashion:

PHYSICS SETTINGS

Boundary Conditions

1 Select the **Interior boundaries** check box in the **Boundary Settings** dialog box.

2 Enter these boundary coefficients:

SETTINGS	BOUNDARIES 5, 9	BOUNDARIES 6, 8
Type	Surface current	Surface current
J_{sz}	`Mpre`	`-Mpre`

Subdomain Settings

Modify the subdomain settings to remove the magnetization:

SETTING	SUBDOMAINS 2, 3
M_x	0

COMPUTING THE SOLUTION

Go to the **Solver Parameters** dialog box and clear the **Adaptive mesh refinement** check box to turn off the adaptive solver. Simply use the mesh obtained by solving the previous problem. Solve the problem again.

POSTPROCESSING AND VISUALIZATION

The solution plot reveals a result identical to the one where you modeled the premagnetization using a magnetization vector.

References

1. D.K. Cheng, *Field and Wave Electromagnetics*. Addison-Wesley. Reading, MA, 1989.

2. J. Jin, *The Finite Element Method in Electromagnetics*, John Wiley & Sons, New York, 1993.

3. B.D. Popovic, *Introductory Engineering Electromagnetics*, Addison-Wesley, Reading, MA, 1971.

6

Fluid Mechanics

This chapter explains how to use the Incompressible Navier-Stokes application mode for the modeling and simulation of fluid mechanics and fluid statics. Note that the engineering community often uses the term *CFD, computational fluid dynamics,* to refer to the numerical simulation of fluids. This chapter concludes with step-by-step instructions on how to model a common benchmark problem: flow over a backward step in the absence of external forces.

The Navier-Stokes Application Mode

Fluid mechanics deals with studies of gases and liquids either in motion (*fluid dynamics*) or at rest (*fluid statics)*. When studying liquid flows, it is often safe to assume that the material's density is constant or almost constant. You then have an *incompressible fluid flow.* Using the Incompressible Navier-Stokes application mode you can solve transient and steady-state models of incompressible fluid dynamics.

In the Model Navigator you find two entry points for **Incompressible Navier-Stokes**. These are **Steady-state analysis**, and **Transient analysis**.

Note: The optional Chemical Engineering Module contains more extensive application modes for incompressible Navier-Stokes problems including non-Newtonian flow and turbulence modeling using the k-ε and k-ω models. It also supplies application modes for non-isothermal and weakly compressible flow, swirl flow, multiphase flow, and the Brinkman equations and Darcy's law for flow in porous media.

Variables and Space Dimension

The Incompressible Navier-Stokes application mode solves for the pressure p and the velocity vector components. It is available for 2D, 2D axisymmetric, and 3D geometries.

PDE Formulation and Equations

COMSOL Multiphysics uses a generalized version of the Navier-Stokes equations to allow for variable viscosity.

Starting with the momentum balance in terms of stresses, the generalized equations in terms of transport properties and velocity gradients are

$$\begin{aligned} \rho\frac{\partial \mathbf{u}}{\partial t} - \nabla \cdot [\eta(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)] + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{F} \\ \nabla\cdot\mathbf{u} &= 0 \end{aligned} \qquad (6\text{-}1)$$

The first equation is the *momentum transport equations*, and the second is the *equation of continuity* for incompressible fluids. The following variables and parameters appear in the equations:

- η is the dynamic viscosity
- ρ is the density
- $\mathbf{u}$ is the velocity field
- p is the pressure
- $\mathbf{F}$ is a volume force field such as gravity

These application modes are general enough to account for all types of incompressible flow. In practice, though, successful analysis of turbulent flows requires simplifications of the description of transport of momentum.

Subdomain Settings

The subdomain quantities are:

PARAMETER	VARIABLE	DESCRIPTION
ρ	`rho`	Density
η	`eta`	Dynamic viscosity
$\mathbf{F}$	`F`	Volume force

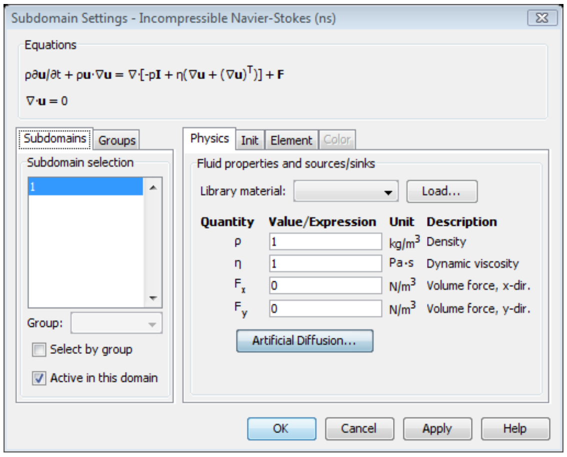
Density This material property specifies the fluid density.

Dynamic Viscosity This term describes the relationship between the shear stresses in a fluid and the shear rate. Intuitively, water and air have a low viscosity, and substances often described as thick, such as oil, have a higher viscosity. You can describe some non-Newtonian fluid by defining a shear-rate dependent viscosity. Examples of non-Newtonian fluids include yoghurt, paper pulp, and polymer suspensions.

Note: The Chemical Engineering Module contains an application mode for non-Newtonian fluids with predefined viscosity models.

Volume Force The volume force vector, $\mathbf{F} = (F_x, F_y, F_z)$, describes a distributed force field such as gravity. The unit of the volume force is force/volume.

You specify the subdomain properties in the **Subdomain Settings** dialog box.


Subdomain Settings - Incompressible Navier-Stokes (ns)

Equations

$\rho\partial\mathbf{u}/\partial t + \rho\mathbf{u}\cdot\nabla\mathbf{u} = \nabla\cdot[-p\mathbf{I} + \eta(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)] + \mathbf{F}$

$\nabla\cdot\mathbf{u} = 0$

Subdomain selection: 1

Group:

☐ Select by group

☑ Active in this domain

Fluid properties and sources/sinks

Library material:

Quantity	Value/Expression	Unit	Description
ρ	1	kg/m^3	Density
η	1	Pa·s	Dynamic viscosity
F_x	0	N/m^3	Volume force, x-dir.
F_y	0	N/m^3	Volume force, y-dir.

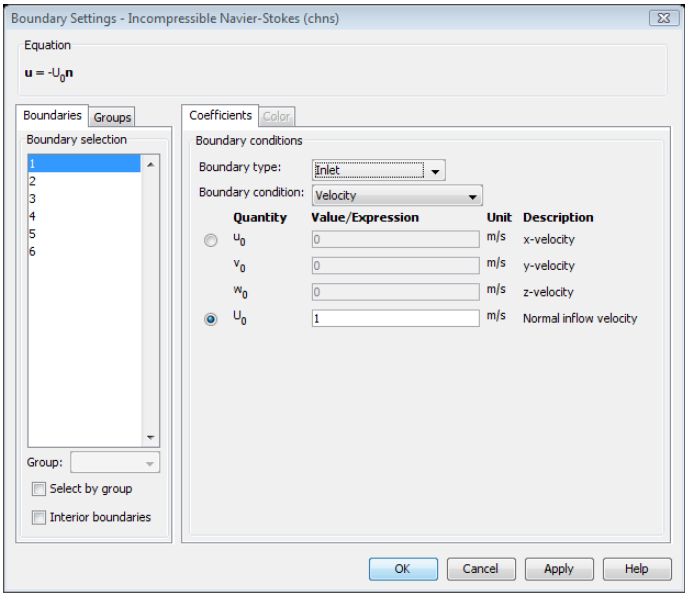
Boundary Conditions

The boundary conditions for the Incompressible Navier-Stokes application mode are grouped into the following types:

- Wall
  - No slip (Default)
  - Slip
  - Sliding wall
  - Moving/leaking wall
- Inlet
  - Velocity (Default)
  - Pressure, no viscous stress
  - Laminar inflow

- Outlet
  - Velocity
  - Pressure
  - Pressure, no viscous stress (Default)
  - No viscous stress
  - Normal stress
  - Laminar outflow
- Symmetry boundary
  - Symmetry (Default)
  - Axial symmetry
- Open boundary
  - Normal stress (Default)
  - No viscous stress
- Stress
  - General stress (Default)
  - Normal stress
  - Normal stress, normal flow

You specify a boundary condition in the **Boundary Settings** dialog where you first select the appropriate **Boundary type** and then a **Boundary Condition**.


Boundary Settings - Incompressible Navier-Stokes (chns)

Equation

$\mathbf{u} = -U_0\mathbf{n}$

Boundary selection: 1, 2, 3, 4, 5, 6

Boundary type: Inlet

Boundary condition: Velocity

	Quantity	Value/Expression	Unit	Description
○	u_0	0	m/s	x-velocity
	v_0	0	m/s	y-velocity
	w_0	0	m/s	z-velocity
●	U_0	1	m/s	Normal inflow velocity

Figure 6-1: Boundary Settings dialog box for the Incompressible Navier-Stokes application mode.

If a mathematical formulation describes more than one type of physical boundary condition, it can appear in more than one boundary type. However, every possible use of a single mathematical formulation cannot be covered. Hence, the boundary types should be regarded as guidelines, not as restrictions on the applicability of the formulations.

The theory of most boundary conditions can be found in Ref. 1.

WALL

These boundary conditions describe the existence of a solid wall.

No Slip

This is the standard and default boundary condition for a stationary solid wall. The condition prescribes

$$\mathbf{u} = \mathbf{0}$$

that is, that the fluid at the wall is not moving.

Moving/Perforated Wall

If the wall moves, so must the fluid. Hence, this boundary condition prescribes

$$\mathbf{u} = \mathbf{u}_{\mathrm{w}}$$

Note that setting this boundary condition does not automatically cause the associated wall to move. The section "The Moving Mesh Application Mode" on page 401 of the *COMSOL Multiphysics Modeling Guide* describes how to set up a model with moving boundaries.

You can also use the Moving/perforated wall boundary condition to simulate a wall where fluid is leaking into or leaving through a perforated wall.

Sliding Wall

If you use this boundary condition, the wall is assumed to behave like a conveyor belt, that is, that the surface is sliding in its tangential direction. The wall does not have to actually move in the coordinate system.

In two space dimensions (2D), the tangential direction is unambiguously defined by the direction of the boundary. However, the situation becomes more complicated in 3D. For this reason, this boundary condition has slightly different definitions in the different space dimensions.

2D and Axial Symmetry The velocity is given as a scalar U_{w} and the condition prescribes

$$\begin{aligned} \mathbf{u} \cdot \mathbf{n} &= 0 \\ \mathbf{u} \cdot \mathbf{t} &= \mathrm{U}_{\mathrm{w}} \end{aligned}$$

where $\mathbf{t} = (-n_y, n_x)$ for 2D and $\mathbf{t} = (-n_z, n_r)$ for axial symmetry.

3D The velocity is set equal to a given vector $\mathbf{u}_{\mathrm{w}}$ projected onto the boundary plane:

$$\mathbf{u} = \mathbf{u}_{\mathrm{w}} - (\mathbf{n} \cdot \mathbf{u}_{\mathrm{w}})\mathbf{n}$$

Slip

The slip condition assumes that there are no viscous effects at the slip wall and hence, no boundary layer develops. From a modeling point of view, this may be a reasonable approximation if the important effect of the wall is to prevent fluid from leaving the domain. Mathematically, the constraint can be formulated as:

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \mathbf{t} \cdot (-p\mathbf{I} + \eta(\nabla\mathbf{u} + (\nabla\mathbf{u})^T))\mathbf{n} = 0$$

where $\mathbf{t}$ is a tangential vector to the boundary.

INLET

This boundary type contains different ways to specify conditions on a boundary where the fluid is supposed to enter the domain. Notice that the formulations contained in this boundary type all appear, some of them slightly modified, in the Outflow boundary type as well. Hence, there is nothing in the mathematical formulations that prevents a fluid from leaving the domain through boundaries where you have specified the Inlet boundary type.

Velocity

This boundary condition offers two ways to specify an inlet velocity. The first is to set the velocity equal to a given vector $\mathbf{u}_0$:

$$\mathbf{u} = \mathbf{u}_0$$

The other is to specify a normal inflow velocity:

$$\mathbf{u} = -\mathbf{n}U_0$$

Note that the boundary normal, $\mathbf{n}$, is pointing out of the domain.

Pressure, No Viscous Stress

This boundary condition specifies vanishing viscous stress along with a Dirichlet condition on the pressure:

$$\eta(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)\mathbf{n} = \mathbf{0}, \qquad p = p_0$$

It is a numerically stable boundary condition that admits total control of the pressure level along the entire boundary. However, if the inflow is not normal to the boundary, this condition is an overspecification. In the case that your solution turns out to have a non-normal inflow velocity, there are two choices. Either, move the boundary farther away to a location where the inflow is normal to the boundary or, use a stress type boundary condition described on page 139.

Note that this condition is identical to the Pressure, no viscous stress condition for Outflow boundaries. Hence, depending on the pressure field in the rest of the subdomain, a boundary with this condition can very well become an outflow boundary.

OUTFLOW

This boundary type contains different ways to specify conditions on a boundary where the fluid exits the domain. Note that all of the formulations in this type can be found, possibly slightly modified, in other boundary types as well. Hence, there is nothing in the mathematical formulations that prevent a fluid from entering the domain through boundaries where you have set the Outflow boundary type.

Setting outlet conditions for the Navier-Stokes equations is not a trivial task. A general rule of thumb, however, is that if there is something interesting happening at an outflow boundary, extend the computational domain to include this phenomenon.

Velocity

This boundary condition offers two ways to specify an outlet velocity. The first is to set the velocity equal to a given vector $\mathbf{u}_0$:

$$\mathbf{u} = \mathbf{u}_0$$

The other is to specify a normal outlet velocity:

$$\mathbf{u} = \mathbf{n}U_0$$

Observe that the boundary normal, $\mathbf{n}$, is pointing out of the domain.

Pressure, No Viscous Stress

This boundary condition specifies vanishing viscous stress along with a Dirichlet condition on the pressure:

$$\eta(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)\mathbf{n} = \mathbf{0}, \qquad p = p_0$$

It is a numerically stable boundary condition that admits total control of the pressure level at the whole boundary. However, if the outflow is not normal to the boundary, this condition is an overspecification. In the case that your solution turns out to have a non-normal outflow velocity, there are two choices. Either move the boundary farther away to a location where the outflow is normal to the boundary or use a stress type boundary condition described on page 139.

Observe that this condition is identical to the Pressure, no viscous stress condition for Inflow boundaries. Hence, depending on the pressure field in the rest of the subdomain, a boundary with this condition can very well become an inflow boundary.

Pressure

This boundary condition prescribes only a Dirichlet condition for the pressure:

$$p = p_0$$

Use this boundary condition only for high Reynolds number outflow boundaries, that is $\mathrm{Re}^c = \rho|\mathbf{u}|h/(2\eta) >> 1$. It is far less stable than the Pressure, no viscous stress boundary condition, but it is consistent with a non-normal outflow velocity.

No Viscous Stress

Prescribes vanishing viscous stress:

$$\eta(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)\mathbf{n} = \mathbf{0}$$

This condition can be useful in some situations because it does not impose any constraint on the pressure. A typical example is a model with volume forces that give rise to pressure gradients that are hard to prescribe in advance. It should however be combined with a point constraint on the pressure to be numerically stable (see “Point Settings” on page 140).

Normal Stress

The total stress on the boundary is set equal to a stress vector of magnitude, f_0, oriented in the negative normal direction:

$$(-p\mathbf{I} + \eta(\nabla\mathbf{u} + (\nabla\mathbf{u})^T))\mathbf{n} = -\mathrm{f}_0\mathbf{n}$$

This implies that the total stress in the tangential direction is zero. This boundary condition implicitly sets a constraint on the pressure that for 2D flows can be written

$$p = 2\eta\frac{\partial u_n}{\partial n} + \mathrm{f}_0 \tag{6-2}$$

If $\partial u_n/\partial n$ is small, Equation 6-2 can be interpreted as $p \approx \mathrm{f}_0$.

SYMMETRY BOUNDARY

Prescribes no penetration and vanishing shear stresses:

$$\mathbf{u}\cdot\mathbf{n} = 0, \quad \mathbf{t}\cdot(-p\mathbf{I} + \eta(\nabla\mathbf{u} + (\nabla\mathbf{u})^T))\mathbf{n} = 0$$

In 2D Axial Symmetry, the above formulation is called **Symmetry**.

Axial Symmetry

This boundary condition is only available in 2D Axial Symmetry. Use it on all boundaries with coordinate $r = 0$. It prescribes $u_r = 0$ and vanishing stresses in the z direction.

OPEN BOUNDARY

You can use this boundary type on boundaries that are open to large volumes of fluid. Fluid can both enter and leave the domain on boundaries with this type of condition.

No Viscous Stress

Prescribes vanishing viscous stress:

$$\eta(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)\mathbf{n} = \mathbf{0}$$

This condition can be useful in some situations because it does not impose any constraint on the pressure. A typical example is a model with volume forces that give rise to pressure gradients that are hard to prescribe in advance. It should however be combined with a point constraint on the pressure to be numerically stable (see "Point Settings" on page 140).

Normal Stress

The total stress on the boundary is set equal to a stress vector of magnitude, f_0, oriented in the negative normal direction:

$$(-p\mathbf{I} + \eta(\nabla\mathbf{u} + (\nabla\mathbf{u})^T))\mathbf{n} = -\mathrm{f}_0\mathbf{n}$$

This implies that the total stress in the tangential direction is zero. This boundary condition implicitly sets a constraint on the pressure that for 2D flows can be written

$$p = 2\eta\frac{\partial u_n}{\partial n} + \mathrm{f}_0 \qquad (6\text{-}3)$$

If $\partial u_n/\partial n$ is small, Equation 6-3 can be interpreted as $p \approx \mathrm{f}_0$.

STRESS

This type of boundary condition represents a very general class of conditions also known as traction boundary conditions.

General Stress

The total stress on the boundary is set equal to a given stress $\mathbf{F}$:

$$(-p\mathbf{I} + \eta(\nabla\mathbf{u} + (\nabla\mathbf{u})^T))\mathbf{n} = \mathbf{F}$$

This boundary condition implicitly sets a constraint on the pressure that for 2D flows can be written

$$p = 2\eta\frac{\partial u_n}{\partial n} - \mathbf{n}\cdot\mathbf{F} \qquad (6\text{-}4)$$

If $\partial u_n/\partial n$ is small, Equation 6-4 can be interpreted as $p \approx -\mathbf{n} \cdot \mathbf{F}$.

Normal Stress

The total stress on the boundary is set equal to a stress vector of magnitude, f_0, oriented in the negative normal direction:

$$(-p\mathbf{I} + \eta(\nabla\mathbf{u} + (\nabla\mathbf{u})^T))\mathbf{n} = -\mathrm{f}_0\mathbf{n}$$

This implies that the total stress in the tangential direction is zero. This boundary condition implicitly sets a constraint on the pressure that for 2D flows can be written

$$p = 2\eta\frac{\partial u_n}{\partial n} + \mathrm{f}_0 \qquad (6\text{-}5)$$

If $\partial u_n/\partial n$ is small, Equation 6-5 can be interpreted as $p \approx \mathrm{f}_0$.

Normal Stress, Normal Flow

In addition to the stress condition set in the Normal stress condition, this condition also prescribes that there must be no tangential velocities on the boundary:

$$(-p\mathbf{I} + \eta(\nabla\mathbf{u} + (\nabla\mathbf{u})^T))\mathbf{n} = -\mathrm{f}_0\mathbf{n}, \quad \mathbf{t} \cdot \mathbf{u} = \mathbf{0}$$

Also this boundary condition implicitly sets a constraint on the pressure that for 2D flows can be written

$$p = 2\eta\frac{\partial u_n}{\partial n} + \mathrm{f}_0 \qquad (6\text{-}6)$$

If $\partial u_n/\partial n$ is small, Equation 6-6 can be interpreted as $p \approx \mathrm{f}_0$.

Point Settings

If it is not possible to specify the pressure level using a boundary condition, the pressure must be set in some other way, for example, by specifying a fixed pressure at a point. You find a dialog box for **Point Settings** on the **Physics** menu.

Numerical Stability—Artificial Diffusion

The momentum equations (Equation 5-7) are of convection-diffusion type. It is well known that if standard Galerkin discretization is used, such equations become unstable for an element Peclet number (Pe) larger than one (Ref. 2):

$$\mathrm{Pe} = \frac{\rho\|\mathbf{u}\|h}{2\eta} > 1 \qquad (6\text{-}7)$$

where h is the element size. The instabilities might cause the simulation to diverge. If a solution where Pe is larger than one is obtained, it can often have spurious oscillations even if the exact solution is smooth. Figure 6-2 shows an example of such ocscillations.

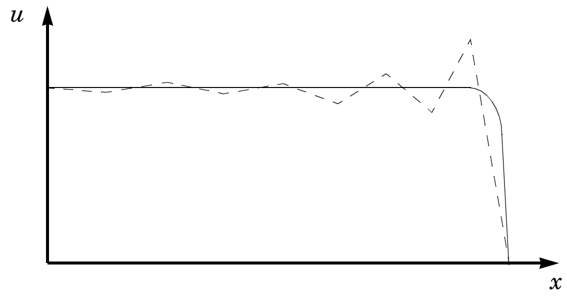
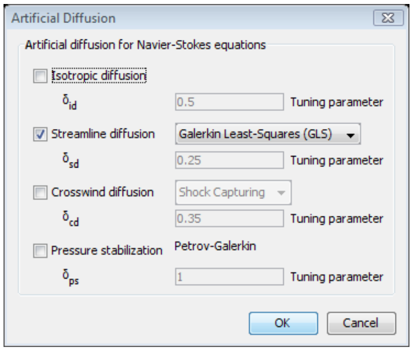


Figure 6-2: A 1D convection diffusion example with $\mathrm{Pe} > 1$. *The exact solution (solid line) is smooth while the solution obtained by standard Galerkin discretization (dashed line) contains spurious oscillations.*

The element Peclet number can always be made less than one by refining the mesh, that is, by making the mesh element size h small enough. This method is not always feasible because it can require a very dense mesh. Instead, so-called stabilization methods, or artificial diffusion, are used. COMSOL Multiphysics provides several of these methods. The dialog box for selecting artificial diffusion (Figure 6-3) is opened from the **Subdomain Settings** dialog box by clicking the **Artificial Diffusion** button (see Figure 5-1 on page 109).


Artificial Diffusion

Artificial diffusion for Navier-Stokes equations

☐ Isotropic diffusion		
δ_{id}	0.5	Tuning parameter
☑ Streamline diffusion	Galerkin Least-Squares (GLS)	
δ_{sd}	0.25	Tuning parameter
☐ Crosswind diffusion	Shock Capturing	
δ_{cd}	0.35	Tuning parameter
☐ Pressure stabilization	Petrov-Galerkin	
δ_{ps}	1	Tuning parameter

Figure 6-3: The Artificial Diffusion dialog box showing the default setting: Galerkin least-squares (GLS) streamline diffusion.

None of the artificial diffusion methods allows arbitrarily high Peclet numbers. As the Peclet number goes to infinity, they either lose their stabilizing effect or completely destroy the solution.

To use no artificial diffusion at all corresponds to the standard Galerkin discretization. A requirement for stability is then that higher-order basis functions must be used for the velocities than for the pressure. This condition is know as the Babuska-Brezzi condition or the inf-sup stability condition (Ref. 1 and 3). Two of the artificial diffusion methods provided in COMSOL Multiphysics can relax the Babuska-Brezzi condition: Galerkin least-squares (GLS) and pressure stabilization.

The mathematical description of the methods is given in Chapter 15, "Stabilization Techniques," in the *COMSOL Multiphysics Modeling Guide*.

ISOTROPIC DIFFUSION

Isotropic diffusion increases the viscosity in regions where the Peclet number is large. Hence, the original equation is perturbed. This method can force the simulation to be stable, but there is no guarantee that the solution is the correct physical solution.

STREAMLINE DIFFUSION

In most cases, spurious oscillations arise in the streamline direction only. Streamline diffusion methods introduce diffusion in the streamline direction by weighting the discretization scheme in the upstream direction. This means that more information is taken from the direction from which information is convected and less from the direction in which the information is traveling. Streamline-diffusion methods are closely related to the upwind methods in finite difference and finite volume methods.

Anisotropic Streamline Diffusion

This method projects the extra diffusion onto the streamlines and is therefore much less brutal than the isotropic diffusion. Still, the method is not consistent, meaning that the exact solution does not solve the discrete system. Therefore, the accuracy of the solution cannot be guaranteed.

Streamline Upwind Petrov-Galerkin (SUPG)

The Streamline Upwind Petrov-Galerkin method is a consistent method, which means that it does not perturb the equations. A model that converges with this method can be considered to be a solution to the discrete counterpart of Equation 5-6 and Equation 5-7. This method is somewhat less stabilizing than the anisotropic streamline diffusion because the terms that differ between the methods are mainly destabilizing.

Streamline Upwind Petrov-Galerkin (SUPG), Compensated

This version of SUPG turns off the diffusion in regions with element Peclet number less than one.

Galerkin Least-Squares (GLS)

The Galerkin least-squares (GLS) method is a consistent method that circumvents the Babuska-Brezzi condition, that is, it is possible to use equal-order elements for the velocities and the pressure.

For the momentum equations, GLS has many of its stability properties in common with SUPG. The superior stability of GLS can often be explained by the stabilizing terms on the continuum equation, terms that SUPG lacks.

GLS is activated per default even though it is more expensive to assemble than SUPG. The extra cost is often more than compensated for by the superior stability properties of GLS.

CROSSWIND DIFFUSION

In some cases, diffusion in the streamline direction is not enough to enforce stability. This is the situation in thin boundary layers and shear layers where there are large gradients in directions orthogonal to the streamline direction.

Ordo $h^{3/2}$

The Ordo $h^{3/2}$ method is an efficient and inexpensive method with good convergence properties. It is not consistent but has good accuracy compared to methods of similar complexity. However, the method is constructed exclusively for linear elements and it is dimensionally inconsistent. The latter is important to note if the model uses other units than SI units.

Shock Capturing

This method is more general and often better than the Ordo $h^{3/2}$ method. It is also consistent and independent of which units that are used. It is, however, considerably more computationally expensive than the Ordo $h^{3/2}$ method.

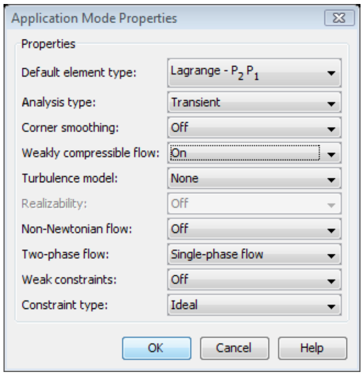
PRESSURE STABILIZATION

The main feature of this method is that it circumvents the Babuska-Brezzi condition. It can also add stability essential to iterative solvers.

Pressure stabilization together with SUPG can be regarded as a less expensive and less stable version of GLS. Pressure stabilization should not be used together with GLS.

Corner Smoothing

You find the **Corner smoothing** property in the **Application Mode Properties** dialog box. It can be a useful property when the model contains walls with slip conditions, as described below.



Application Mode Properties

Properties

Default element type:	Lagrange - P_2 P_1
Analysis type:	Transient
Corner smoothing:	Off
Weakly compressible flow:	On
Turbulence model:	None
Realizability:	Off
Non-Newtonian flow:	Off
Two-phase flow:	Single-phase flow
Weak constraints:	Off
Constraint type:	Ideal

Figure 6-4: The Application Mode Properties dialog box where Corner smoothing can be turned on and off.

Consider the situation sketched in Figure 6-5. At the point where the boundaries Γ_1 and Γ_2 intersect, there will be two boundary normals, one for Γ_1 and one for Γ_2. These two normals are denoted $\mathbf{n}_{\Gamma_1}$ and $\mathbf{n}_{\Gamma_2}$ in Figure 6-5. If the boundaries now both have no-penetration condition, there will be two Dirichlet conditions at the point of intersection, namely

$$\mathbf{n}_{\Gamma_1} \cdot \mathbf{u} = 0 \qquad (6\text{-}8)$$

and

$$\mathbf{n}_{\Gamma_2} \cdot \mathbf{u} = 0 \qquad (6\text{-}9)$$

The only way that both Equation 6-8 and Equation 6-9 can be fulfilled is if $\mathbf{u} \equiv \mathbf{0}$ at the point of intersection. This is not always the expected solution, however.

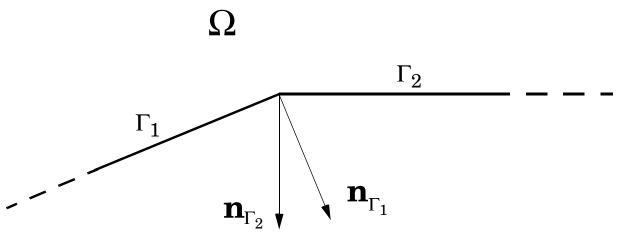


Figure 6-5: Intersection between the boundaries Γ_1 *and* Γ_2*.* $\mathbf{n}_{\Gamma_1}$ *and* $\mathbf{n}_{\Gamma_2}$ *are the boundary normals prescribed by* Γ_1 *and* Γ_2 *respectively.* Ω *is the computational domain.*

When corner smoothing is activated, any Dirichlet condition $d(\mathbf{n}) = 0$ is replaced by $d(\mathbf{n}_\mathrm{w}) = 0$, where $\mathbf{n}_\mathrm{w}$ is a vector of dependent boundary variables whose solution in each point is the average of all normals in that point. In the current example, equations 6-8 and 6-9 are replaced with

$$\mathbf{n}_\mathrm{w} \cdot \mathbf{u} = \mathbf{0} \qquad (6\text{-}10)$$

and $\mathbf{n}_\mathrm{w}$ has the solution $\mathbf{n}_\mathrm{w} = (\mathbf{n}_{\Gamma_1} + \mathbf{n}_{\Gamma_2})/2$. Equation 6-10 can then be satisfied for $\mathbf{u} \neq \mathbf{0}$.

Application Mode Variables

A number of variables and physical quantities are available for postprocessing and for use in equations and boundary conditions. They appear in the following tables (where x_i denotes the various space coordinate directions):

NAME	TYPE	DESCRIPTION	EXPRESSION
u, v, (w)	B S V	x_i velocity	u, v, w
p	B S	Pressure	p

NAME	TYPE	DESCRIPTION	EXPRESSION
U	B S	Velocity field	$\sqrt{\sum_i (u_i)^2}$
V	S	Vorticity	$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ (2D)
Vxi	S	Vorticity	x_i components of $\nabla \times \mathbf{u}$ (3D)
rho	S	Density	ρ
eta	S	Dynamic viscosity	η
F_xi	S	Volume force, x_i dir.	x_i components of $\mathbf{F}$
K_xi	B	Viscous force per area, x_i component	$\sum_j n_j \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$
T_xi	B	Total force per area, x_i component	$\sum_j n_j\left[-p\delta_{ij} + \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]$
cellRe	S	Cell Reynolds number	$\frac{\rho\lvert\mathbf{u}\rvert h}{\eta}$
res_ui_c	S	Equation residual, u_i component	$\rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \mathbf{F} -$ $\nabla\cdot[\eta(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)]$ (u_i component)
res_sc_ui_c	S	Shock capturing residual, u_i component	$\rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \mathbf{F}$ (u_i component)
beta_xi	S	Convective field, x_i component	ρu_i
Dm	S	Mean diffusion coefficient	η
da	S	Total time-scale factor	ρ

Note: To form the complete application mode variable names, add a suffix consisting of an underscore and the application mode name (default: `ns`), for example, `V_ns`. (This does not apply to the dependent variables for the velocities and pressure.)

General Solver Settings

For a flow that is parallel to a coordinate axis, the automatic scaling feature in COMSOL Multiphysics does not work if you use the nonlinear stationary solver. For such cases, turn off the scaling feature or use manual scaling; see "Scaling of Variables and Equations" on page 497 in the *COMSOL Multiphysics Reference Guide*. The problem occurs when one solution component is identically zero.

Solver Settings

The recommended solver type for small and medium-sized fluid-flow problems is a direct solver, and the PARDISO direct solver is the default solver for 1D and 2D models. It is not as robust as UMFPACK or SPOOLES but is more memory efficient.

3D problems are often too large to solve using a direct solver. The default solver is GMRES with the geometric multigrid (GMG) preconditioner. The GMG preconditioner uses the Vanka preconditioner/smoother as the presmoother and postsmoother.

For more information about the solvers, see Chapter 6, "Solving the Model," on page 359 in the *COMSOL Multiphysics User's Guide*.

For a flow that is parallel to a coordinate axis, the automatic scaling feature in COMSOL Multiphysics does not work if you use the nonlinear stationary solver. For such cases, turn off the scaling feature or use manual scaling; see "Scaling of Variables and Equations" on page 497 in the *COMSOL Multiphysics Reference Guide*. The problem occurs when one solution component is close to zero

Khan and Richardson Force for Particle Tracing

The *Khan and Richardson force* is available for particle tracing plots in the Incompressible Navier-Stokes application mode and other application modes for fluid dynamics in the COMSOL Multiphysics products.

BACKGROUND

The force expression that the software uses is derived partly using experimental results, and it is valid for a wide range of Reynolds numbers, stretching from creeping flow toward the turbulent regime (Ref. 4).

The following equation describes the total force that a fluid exerts on an immersed spherical particle:

$$F = \pi r_p^2 \rho(\bar{u} - \bar{u}_p)^2 (1.84(\mathrm{Re}_p)^{-0.31} + 0.293(\mathrm{Re}_p)^{0.06})^{3.45}$$

where the definition of the particle Reynolds number, Re_p, is

$$\mathrm{Re}_p = (|\bar{u} - \bar{u}_p| 2 r_p \rho)/\eta \, .$$

The orientation of a given force component (for example, positive or negative x-component) is determined by the sign of the corresponding component in the vector difference $\bar{u} - \bar{u}_p$, because this determines whether the fluid is accelerating or slowing down the particle in that direction.

USING THE KHAN AND RICHARDSON FORCE FOR PARTICLE TRACING

Thew Khan and Richardson force is the default selection in the **Predefined forces** list on the **Particle Tracing** tab.

There is one parameter in this force expression: the particle radius r_p, which you defined by clicking the **Parameters** button. Table 6-1 shows the default values for this parameter:

TABLE 6-1: PARAMETER FOR THE KHAN AND RICHARDSON FORCE

PARAMETER NAME IN EQUATION	DESCRIPTION	DEFAULT VALUE	DEFAULT VARIABLE NAME
r_p	particle radius	10^{-4} m	partrad

The particle mass appears in the force equation, and you enter its value in the **m** edit field.The default value for the particle mass is $m_p = (4\pi/3)\cdot 10^{-9}$ kg, which is the mass of a particle with the same density as water and a radius of 10^{-4} m, which is the default radius.

Note: The default settings work for models that use SI units.

References

1. P.M. Gresho and R.L. Sani, *Incompressible Flow and the Finite Element Method, Volume 2: Isothermal Laminar Flow*, John Wiley and Sons, LTD, 2000.

2. O.C. Zienkiewicz, R.L. Taylor, and P. Nithiarasu, *The Finite Element Method for Fluid Dynamics*, Sixth edition, Elsevier, 2005.

3. C. Johnson, *Numerical solution of partial differential equations by the finite element method*, Studentlitteratur, 1987.

4. J.M. Coulson and J.F. Richardson, “Particle Technology and Separation Processes,” *Chemical Engineering*, vol. 2, Butterworth-Heinemann.

Example: Steady Incompressible Flow

Introduction

This model examines the physics of plane, incompressible, and steady flow: flow over a backward step in the absence of external forces. This is a common benchmark problem in CFD. There is no known exact solution, but experimental data has been published (see Ref. 1) making it possible to check the accuracy of the FEM solution. The model includes analyses using both regular triangular meshes and mapped meshes, comparing the solutions for various mesh densities.

Model Definition

Fluid enters from the left side with a parabolic velocity profile, passes over a step, and leaves through the right boundary (Figure 6-6 shows the model geometry).

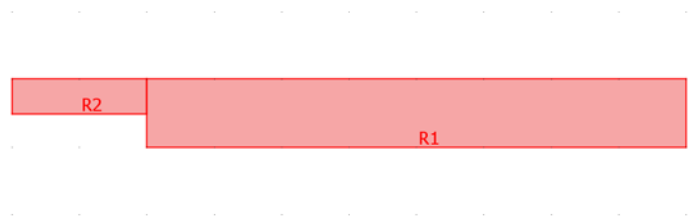


Figure 6-6: The backstep geometry.

The model computes the fluid's velocity components $\mathbf{u} = (u, v)$ in the x and y directions and its pressure p in the region defined by the geometry in the preceding figure. The PDE model for this application uses the stationary incompressible Navier-Stokes equations

$$\begin{cases} -\eta\nabla^2\mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{F} \\ \nabla\cdot\mathbf{u} = 0 \end{cases}$$

where the quantities are (using SI units):

- Dynamic viscosity, $\eta = 1.79\cdot 10^{-5}$ Pa·s

- Density, $\rho = 1.23\ \mathrm{kg/m^3}$
- A force field, **F**, absent in this model

The first equation is the balance of momentum from Newton's second law. The other relationship is the equation of continuity, where zero on the right-hand side states that the fluid is incompressible.

The shape of the flow pattern depends only on the Reynolds number.

In this model, you choose boundary conditions so that the average velocity at the inlet is $V_{\mathrm{mean}} = 0.544\ \mathrm{m/s}$. To obtain a corresponding parabolic velocity profile, set $(u, v) = (6V_{\mathrm{mean}}s(1-s), 0)$, where s is a boundary parameterization variable that runs from 0 to 1 along the boundary. The fluid is always stationary at the walls, so $(u, v) = (0, 0)$ is the appropriate boundary condition. At the exit boundary, assume a constant static pressure $p = 0$.

For such a fluid flow you can expect a velocity field with a boundary layer of thickness approximately equal to $1/\sqrt{\mathrm{Re}}$ at the walls. To resolve this steep solution gradient you need a few rows of elements across the layer. For a flow with a large Reynolds number, elements in the interior of the channel can be much larger than those near the walls.

USING THE INCOMPRESSIBLE NAVIER-STOKES APPLICATION MODE

Start by setting up a model and solve this problem on a fixed isotropic mesh. The given input data correspond to a Reynolds number of

$$\mathrm{Re} = \frac{0.544 \cdot 2 \cdot 0.0052 \cdot 1.23}{1.79 \cdot 10^{-5}} \approx 389$$

Even though you are working with a steady flow model, it needs initial conditions because the incompressible Navier-Stokes equations are nonlinear. To achieve a numerical solution, the nonlinear solver solves the equations iteratively.

Results

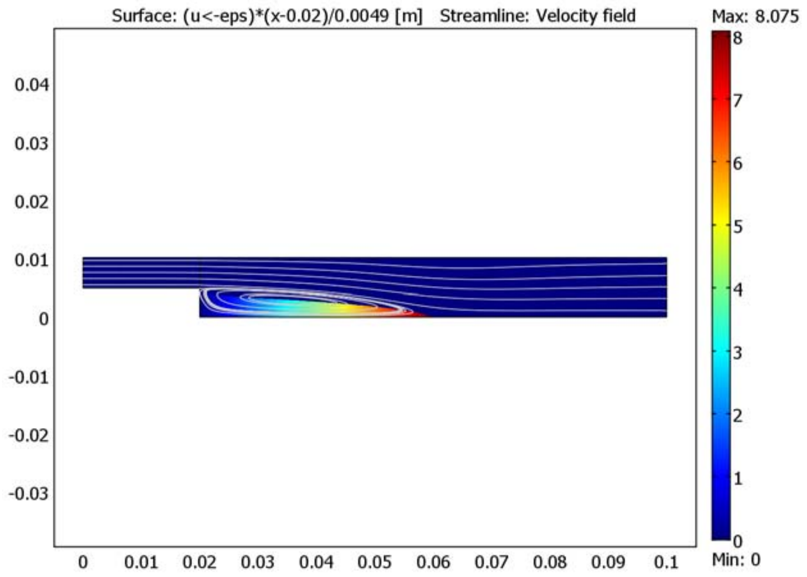


Figure 6-7: A streamline plot of the velocity field.

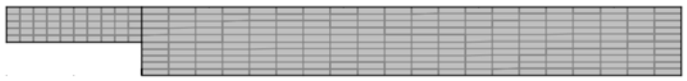
Figure 6-7 shows the velocity field in a streamline plot. Behind the step you can identify a recirculation region that expands with increasing Re. The distance from the step to the *stagnation point*, where the flow reattaches to the lower wall, is the *reattachment length*. A dimensionless number quantifying the recirculation region is the reattachment length divided by the step height. According to experimental data, this quotient is approximately 7.93 at a Reynolds number of 389.

For a high enough Re, the eddy crosses the outflow boundary, which means that there is an inflow through the assumed outflow boundary. This finding calls for the inclusion of more of the downstream region in the computational domain to get a reliable solution. That fact explains why the computational domain in this model is slightly longer than the one in Ref. 2.

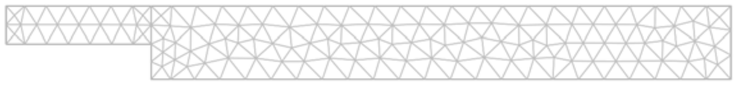
The Incompressible Navier-Stokes application mode uses Lagrange p2-p1 elements to stabilize the pressure. Thus 2nd-order Lagrange elements model the velocity components while linear elements model the pressure. The default element settings in this application mode always provide one order higher Lagrange elements for the velocity components than for the pressure. Stabilizing the solution using streamline diffusion is not recommended for the Navier-Stokes equations except at higher Reynolds numbers.

Four mesh cases were considered for this model: a homogeneous structured mesh, a homogeneous unstructured mesh, and two inhomogeneous mesh cases, where local mesh refinements provide improved accuracy around the point where the step is taken. The following plots show examples of the mesh cases:

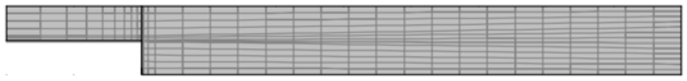
Case 1: Homogeneous structured mesh



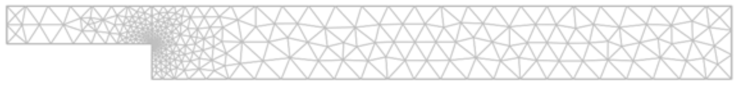
Case 2: Homogeneous unstructured mesh



Case 3: Structured mesh



Case 4: Unstructured mesh



To compare the results using these mesh cases, we solved the model for varying mesh densities for all four mesh cases. The graph in Figure 6-8 shows the resulting reattachment length divided by the step height as a function of the degrees of freedom for the different mesh cases:

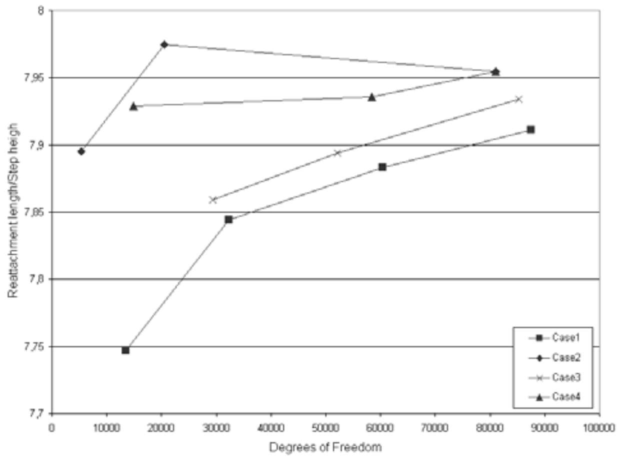


Figure 6-8: Reattachment length divided by step height as functions of the degrees of freedom for Case 1 (squares), Case 2 (diamonds), Case 3 (x-marks), and Case 4 (triangles).

All solutions give reasonable results. Comparing results for structured and unstructured meshes and different numbers of degrees of freedom (DOFs), it is evident that the unstructured mesh cases give more accurate and more stable results even with a lower number of DOFs. However, in the limit of an infinite number of DOFs, the results for all mesh cases converge to the experimental result of 7.93.

The number of mesh elements is related to the number of DOFs. However, the relation is not the same for structured and unstructured meshes. The same number of mesh elements create more DOFs for a structured mesh than for an unstructured mesh, simply because of the rectangular shape of the mesh elements.

For the same number of DOFs the solution time is longer for a structured mesh than for an unstructured mesh. This is due to the structured mesh having a stronger coupled system, which results in system matrices that are less sparse.

References

1. P.M. Gresho and R.L. Sani, *Incompressible Flow and the Finite Element Method*, Volume 1 & 2, John Wiley & Sons, New York, 2000.

2. A. Rose and B. Simpson: “Laminar, Constant-Temperature Flow Over a Backward Facing Step,” *1st NAFEMS Workbook of CFD Examples*, Glasgow, UK, 2000.

Model Library path: `COMSOL_Multiphysics/Fluid_Dynamics/backstep`

Model Library path: `COMSOL_Multiphysics/Fluid_Dynamics/backstep_quad`

Modeling Using the Graphical User Interface

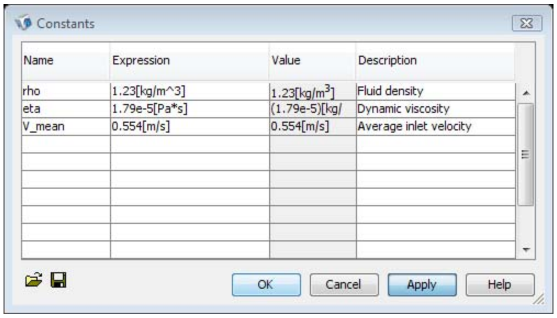
MODEL NAVIGATOR

1. Go to the **Model Navigator** and select **2D** in the **Space dimension** list.
2. In the list of application modes open the **COMSOL Multiphysics>Fluid Dynamics** folder and then the **Incompressible Navier-Stokes** folder. Select **Steady-state analysis**.
3. Make sure that **Lagrange -** $\mathbf{P_2\ P_1}$ is the element type in the **Element** list.
4. Click **OK**.

OPTIONS AND SETTINGS

1. To parameterize the model, go to the **Options** menu and choose **Constants**.
2. Make the following entries in the **Constants** dialog box to represent the fluid properties and velocity (the descriptions are optional):

NAME	EXPRESSION	DESCRIPTION
`rho`	`1.23[kg/m^3]`	`Fluid density`
`eta`	`1.79e-5[Pa*s]`	`Dynamic viscosity`
`V_mean`	`0.554[m/s]`	`Average inlet velocity`



Name	Expression	Value	Description
rho	1.23[kg/m^3]	1.23[kg/m^3]	Fluid density
eta	1.79e-5[Pa*s]	(1.79e-5)[kg/	Dynamic viscosity
V_mean	0.554[m/s]	0.554[m/s]	Average inlet velocity

3 Click **OK** to close the dialog box.

GEOMETRY MODELING

1 Shift-click the **Rectangle/Square** button to specify a rectangle.

2 In the **Rectangle** dialog box, type `0.08` in the **Width** edit field and `0.0101` in the **Height** edit field. In the **Position** area, type `0.02` in the **x** edit field (keep the **Base** setting **Corner**).

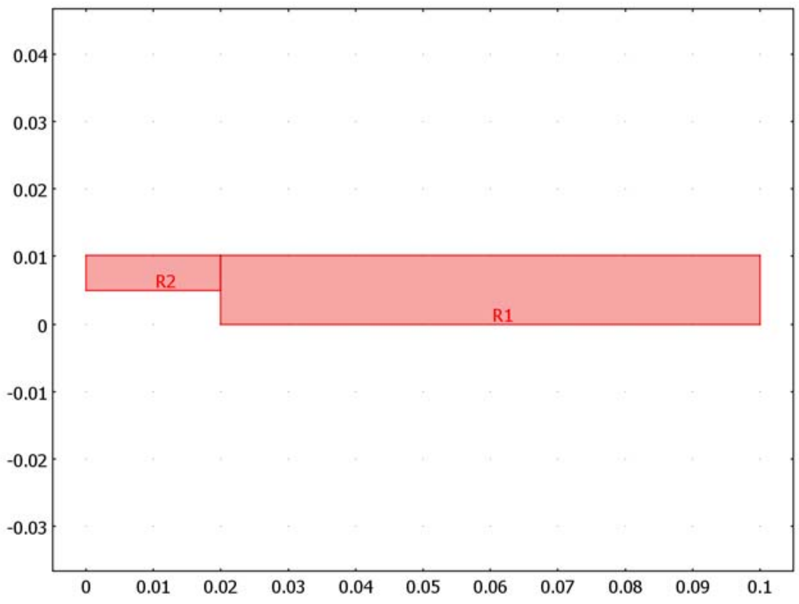
3 Click **OK**.

4 Shift-click the **Rectangle/Square** button to specify another rectangle.

5 In the **Rectangle** dialog box, type `0.02` in the **Width** edit field and `0.0052` in the **Height** edit field. In the **Position** area, type `0.0049` in the **y** edit field (again, keep the **Base** setting **Corner**).

6 Click **OK**.

7 Click the **Zoom Extents** button.



PHYSICS SETTINGS

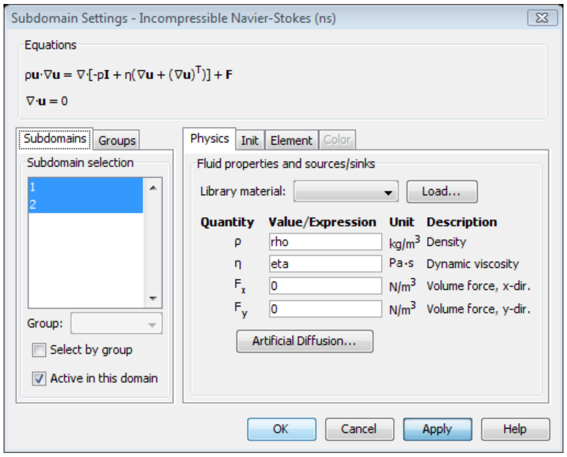
Subdomain Settings

With appropriate scaling it is possible to set the fluid density $\rho = 1$ and the viscosity $\eta = 1/\mathrm{Re}$. If you model in this way, the Reynolds number contains information about fluid density together with details on length scale, velocity scale, and viscosity. For such a model, type `1/Re` in the **Dynamic viscosity** edit field in the **Subdomain Settings** dialog box. Be aware, though, that you must reverse the scaling before interpreting the results quantitatively.

This model, however, uses actual fluid properties specified in SI units.

1. From the **Physics** menu choose **Subdomain Settings** to launch the **Subdomain Settings** dialog box.
2. Select Subdomains 1 and 2.
3. In the **Density** edit field type `rho`.

4 In the **Dynamic viscosity** edit field type `eta`.


Subdomain Settings - Incompressible Navier-Stokes (ns)

Equations

$\rho\mathbf{u}\cdot\nabla\mathbf{u} = \nabla\cdot[-p\mathbf{I} + \eta(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)] + \mathbf{F}$

$\nabla\cdot\mathbf{u} = 0$

Subdomain selection: 1, 2

Fluid properties and sources/sinks

Library material:

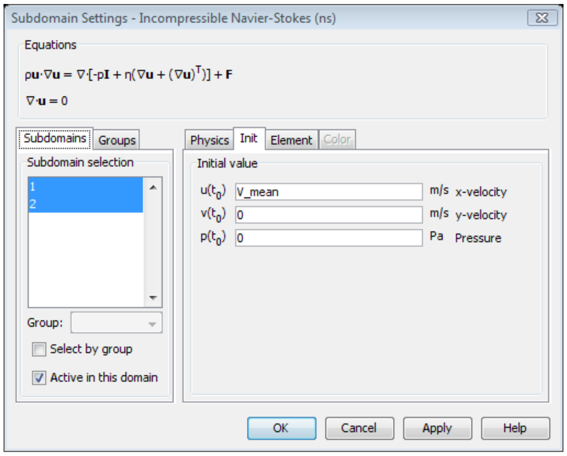
Quantity	Value/Expression	Unit	Description
ρ	rho	kg/m^3	Density
η	eta	Pa·s	Dynamic viscosity
F_x	0	N/m^3	Volume force, x-dir.
F_y	0	N/m^3	Volume force, y-dir.

[ ] Select by group
[✓] Active in this domain

The nonlinear solver also requires an initial value. With a low Reynolds number a simple constant expression is sufficient:

5 With both subdomains still selected, click the **Init** tab.

6 Enter the initial value `V_mean` in the edit field for **u(t$_0$)**; this value corresponds to a uniform *x*-velocity of magnitude V_{mean} throughout the domain. Leave the initial values for the *y*-velocity and the pressure at their default zero values.


Subdomain Settings - Incompressible Navier-Stokes (ns)

Equations

$\rho\mathbf{u}\cdot\nabla\mathbf{u} = \nabla\cdot[-p\mathbf{I} + \eta(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)] + \mathbf{F}$

$\nabla\cdot\mathbf{u} = 0$

Subdomain selection: 1, 2

Initial value

$u(t_0)$	V_mean	m/s	x-velocity
$v(t_0)$	0	m/s	y-velocity
$p(t_0)$	0	Pa	Pressure

[ ] Select by group
[✓] Active in this domain

7 Click **OK**.

Boundary Conditions

The relevant boundary conditions for the model are:

SETTINGS	BOUNDARY 1	BOUNDARIES 2–5, 7	BOUNDARY 8
Boundary type	Inlet	Wall	Outlet
Boundary condition	Velocity	No slip	Pressure, no viscous stress
u_0	`V_mean*6*s*(1-s)`		
v_0	0		
p_0			0

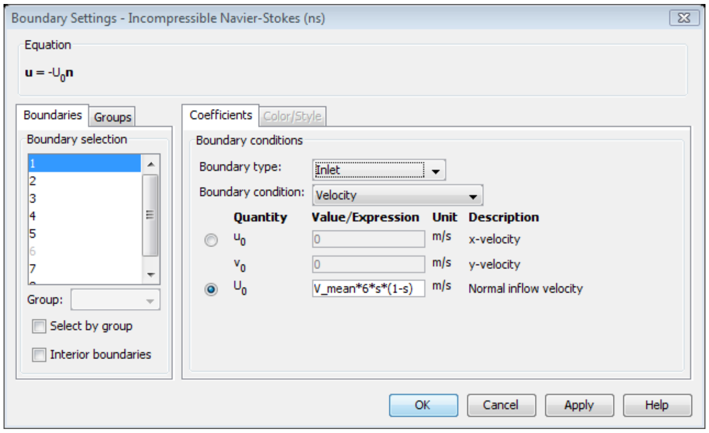
The no-slip condition is the default setting, so you only need to modify the settings for Boundaries 1 and 8.

1 From the **Physics** menu, choose **Boundary Settings**.

2 Select Boundary 1.

3 In the **Boundary type** list select **Inlet**

4 In the $\mathbf{U_0}$ edit field type `V_mean*6*s*(1-s)`.


Boundary Settings - Incompressible Navier-Stokes (ns)

Equation

$\mathbf{u} = -U_0\mathbf{n}$

Boundary selection: 1 (selected), 2, 3, 4, 5, 6, 7

Boundary conditions

Boundary type: Inlet

Boundary condition: Velocity

Quantity	Value/Expression	Unit	Description
u_0	0	m/s	x-velocity
v_0	0	m/s	y-velocity
● U_0	V_mean*6*s*(1-s)	m/s	Normal inflow velocity

5 Select Boundary 8.

6 In the **Boundary type** list select **Outlet**.

7 Click **OK**.

MESH GENERATION

Because the Navier-Stokes equations are computationally difficult, it is important to use an appropriate mesh. If the mesh is too coarse, the solution might not converge at all or errors might be large. Conversely, if the mesh is too fine, the solution time for the nonlinear system of equations might be unnecessarily long. In this case, solve the model with two different types of mesh, first using a mapped mesh and later using a conventional unstructured triangular mesh.

Mesh Case 1—Mapped mesh

1 From the **Mesh** menu, choose **Mapped Mesh Parameters**.

2 In the **Mapped Mesh Parameters** dialog box, click the **Boundary** tab.

3 Specify the numbers of mesh elements according to the following table by first selecting the appropriate boundaries in the **Boundary selection** list, then selecting the **Constrained edge element distribution** check box, and finally entering the number of elements in the **Number of edge elements** edit field.

SETTINGS	BOUNDARIES 1–4, 6	BOUNDARIES 5, 7
Number of mesh elements	34	62

4 Click **Remesh**, then click **OK**.

Mesh Case 2—Triangular mesh

1 From the **Mesh** menu, choose **Free Mesh Parameters**.

2 On the **Global** page, click the **Custom mesh size** option button, then type `9e-4` in the **Maximum element size** edit field.

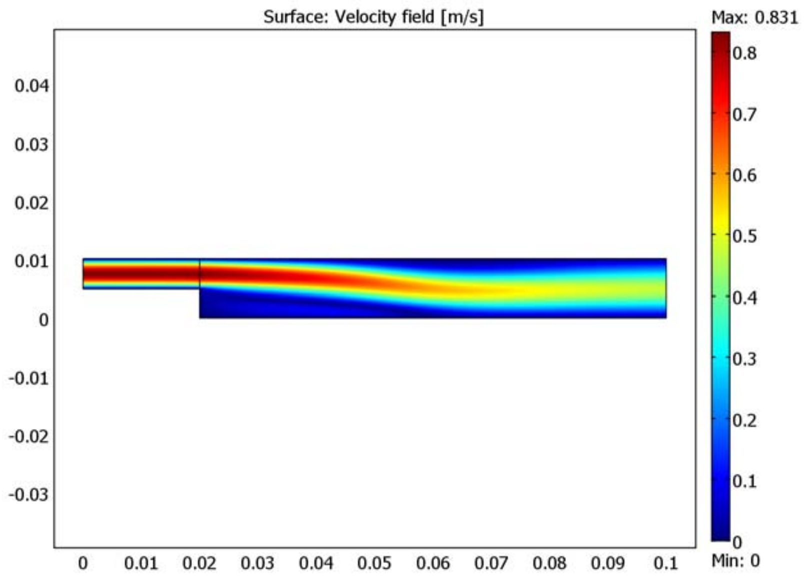
3 Click **Remesh**, then click **OK**.

COMPUTING THE SOLUTION

Click the **Solve** button on the Main toolbar to start the simulation.

POSTPROCESSING AND VISUALIZATION

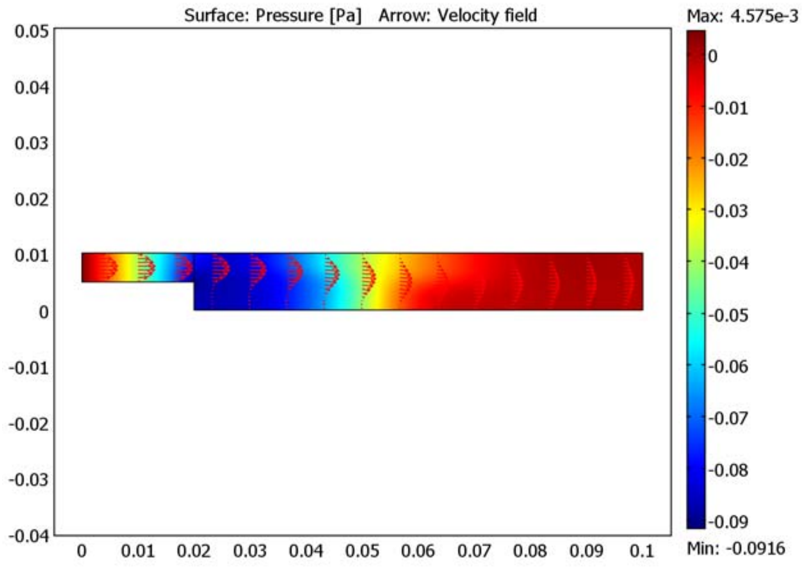
The default visualization displays the magnitude of the velocity field.



To see the velocity field and the pressure field simultaneously, use a combination of arrow and surface plots:

1 From the **Postprocessing** menu, choose **Plot Parameters** to open the **Plot Parameters** dialog box.
2 Click the **Surface** tab.
3 On the **Surface Data** page, select **Incompressible Navier-Stokes (ns)>Pressure** from the **Predefined quantities** list.
4 Click the **Arrow** tab.
5 Select the **Arrow plot** check box.

6 Click **OK** to view the combined plot.



To reproduce the plot in Figure 6-7, visualizing the recirculation region, use a conditional expression testing for a negative x-component in the velocity.

1 Open the **Plot Parameters** dialog box again. Add a streamline plot and remove the arrows by selecting the **Streamline** check box and clearing the **Arrow** check box on the **General** page.

2 Click the **Surface** tab.

3 On the **Surface Data** page, type `(u<-eps)*(x-0.02)/0.0049` in the **Expression** edit field. Clear the **Smooth** check box (the given expression is deliberately discontinuous).

4 Click the **Streamline** tab.

5 On the **Start Points** page, click the **Specify start point coordinates** button.

6 In the **x** edit field type `linspace(0.03,0.03,9)` and in the **y** edit field type `linspace(5e-4,95e-4,9)`. These values result in nine streamlines, which at $x = 0.03$ m are equally spaced in the y direction.

7 Click the **Line Color** tab, then click the **Color** button.

8 In the **Streamline Color** dialog box, click the lightest gray swatch, then click **OK**.

9 Click **OK** to close the **Plot Parameters** dialog box and display the new plot.

7

Heat Transfer

This chapter covers heat transfer application modes. It starts with some background on heat transfer. It then reviews specifics of the Conduction application mode and then the Convection and Conduction application mode. It concludes with COMSOL Multiphysics models of three heat-transfer examples taken from a NAFEMS (National Agency for Finite Element Methods and Standards) benchmark collection.

Heat Transfer Fundamentals

What is Heat Transfer?

From the kitchen toaster to the latest high-performance microprocessor, heat is ubiquitous and of great importance in the engineering world. To optimize thermal performance and reduce costs, engineers and researchers are making use of finite element analysis. Because most material properties are temperature-dependent, the effects of heat enter many other disciplines and drive the requirement for multiphysics modeling.

For instance, both the toaster and the microchip contain electrical conductors that generate thermal energy as electric current passes through them. As these conductors release thermal energy, system temperature increases as does that of the conductors. If the electric conductivity is temperature dependent, it changes accordingly and, in turn, affects the electric field in the conductor. Other examples of multiphysics couplings that involve heat transfer are thermal stresses, thermal-fluid convection, and induction heating.

Heat transfer is defined as the movement of energy due to a temperature difference. It is characterized by the following three mechanisms:

- *Conduction* is heat transfer by diffusion in a stationary medium due to a temperature gradient. The medium can be a solid or a liquid.
- *Convection* is heat transfer between either a hot surface and a cold moving fluid or a cold surface and a hot moving fluid. Convection occurs in fluids (liquids and gases).
- *Radiation* is heat transfer via electromagnetic waves between two surfaces (A and B) with different temperatures T_A and T_B, provided that Surface A is visible to an infinitesimally small observer on Surface B.

Note: The Heat Transfer Module supports simulations of all three types of heat transfer mechanism, including surface-to-surface and surface-to-ambient radiation.

The examples later in this chapter shows transient heat transfer by conduction, convection, and radiation. For an introductory example of a multiphysics coupling of a heat balance to a momentum balance through the Navier-Stokes equations, see “Free

Convection" on page 357 in the *COMSOL Multiphysics Model Library.* In that model the heat flux accounts for transport by convection and conduction. The system consists of heated tubes subjected to a stream of a fluid perpendicular to the main axis of the tubes. The tubes heat the streaming medium as it travels from the bottom to the top of the domain.

The Heat Equation

The mathematical model for heat transfer by conduction is the *heat equation*:

$$\rho C\frac{\partial T}{\partial t} - \nabla \cdot (k \nabla T) = Q$$

Quickly review the variables and quantities in this equation:

- T is the temperature.
- ρ is the density.
- C is the *heat capacity.*
  - C_p is the heat capacity at a constant pressure.
  - C_v is the heat capacity for a constant volume.
- k is *thermal conductivity.*
- Q is a *heat source* or a *heat sink*.

For a steady-state model, temperature does not change with time, and the first term containing ρ and C vanishes.

If the thermal conductivity is anisotropic, k becomes the thermal conductivity tensor k:

$$k = \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix}$$

To model heat conduction and convection through a fluid, the heat equation also includes a convective term. COMSOL Multiphysics represents this formulation in the Convection and Conduction application mode as:

$$\rho C_p \frac{\partial T}{\partial t} + \nabla \cdot (-k \nabla T + \rho C_p T \mathbf{u}) = Q$$

where **u** is the velocity field. This field can either be provided as a mathematical expression of the independent variables or calculated by a coupling to the velocity field from an application mode such as Incompressible Navier-Stokes.

The heat flux vector is defined by the expression within the parentheses in the equation above. For transport through conduction and convection, this equation yields

$$\mathbf{q} = -k\nabla T + \rho C_p T \mathbf{u}$$

where **q** is the heat flux vector. If the heat transfer is by conduction only, **q** is determined by

$$\mathbf{q} = -k\nabla T$$

For a detailed discussion of the fundamentals of heat transfer, see Ref. 1.

Note: *Heat capacity* here refers to the quantity that represents the amount of heat required to change one unit of mass of a substance by one degree. It has units of energy per mass per degree (J/kg/K in SI units). This quantity is also called *specific heat* or *specific heat capacity*.

The Conduction Application Mode

This application mode models heat transfer by conduction. It also includes convection and radiation effects around edges and boundaries. The Conduction application mode is suitable for modeling of heat transfer in solids. To start modeling with this application mode, go to the **Model Navigator** and select **Heat Transfer** and then **Conduction**.

Variables and Space Dimensions

The Conduction application mode is available in 1D, 2D, and 3D as well as for axisymmetric models using cylindrical coordinates in 1D and 2D. The dependent variable is the temperature, T.

PDE Formulation

The mathematical model for heat transfer by conduction is the following version of the *heat equation*

$$\delta_{\mathrm{ts}}\rho C_p\frac{\partial T}{\partial t}-\nabla\cdot(k\nabla T) = Q$$

with the following material properties:

- δ_{ts} is a *time-scaling coefficient*.
- ρ is the *density*.
- C_p is the *heat capacity*.
- k is the *thermal conductivity* tensor.
- Q is the heat source (or sink).

For a steady-state problem the temperature does not change with time and the first term disappears.

In rare cases you might decide to add domain-specific transversal convection or radiation in 1D planar and axisymmetric models and 2D planar models. Represent this by adding the two terms on the right:

$$\delta_{\mathrm{ts}}\rho C_p\frac{\partial T}{\partial t}-\nabla\cdot(\underline{k}\nabla T) = Q+\frac{h_{\mathrm{trans}}}{dA}(T_{\mathrm{ext}}-T)+\frac{C_{\mathrm{trans}}}{dA}(T_{\mathrm{ambtrans}}^4-T^4)$$

where:

- h_{trans} is the *transversal convective heat transfer coefficient*.
- T_{ext} is the *transversal external temperature*.
- C_{trans} is a *user-defined constant*.
- T_{ambtrans} is the *transversal ambient temperature*.
- dA is the thickness in 2D and area in 1D.

Note: When using transversal convection or radiation, the heat equation must be modified to accommodate the area dA. For all practical purposes the use of this feature is a rare exception, and the terms h_{trans} and C_{trans} are usually set to zero.

Subdomain Settings

The subdomain quantities are:

QUANTITY	VARIABLE	DESCRIPTION
δ_{ts}	`Dts`	Time-scaling coefficient
ρ	`rho`	Density
C_p	`C`	Heat capacity
k	`k`	Thermal conductivity
$\underline{k}$		Thermal conductivity tensor
k_{ij}	`kx`*`i`*`x`*`j`*	Thermal conductivity tensor, x_ix_j component
Q	`Q`	Heat source
h_{trans}	`htrans`	Transversal convective heat transfer coefficient
T_{ext}	`Text`	Transversal external temperature
C_{trans}	`Ctrans`	User-defined constant
T_{ambtrans}	`Tambtrans`	Transversal ambient temperature

Time-scaling coefficient This coefficient is normally 1, but you can change the time scale, for example, from seconds to minutes by setting it to 1/60.

Density It specifies the material's density. Enter this quantity as mass per volume.

Heat capacity The heat capacity C describes the amount of heat energy required to produce a unit temperature change in a unit mass.

Thermal conductivity The thermal conductivity k describes the relationship between the heat flux vector $\mathbf{q}$ and the temperature gradient ∇T as in

$$\mathbf{q} = -k\nabla T,$$

which is *Fourier's law of heat conduction*. Enter this quantity as power per length and temperature.

Heat source The heat source describes heat generation within the domain. Express heating and cooling with positive and negative values, respectively. Enter the quantity Q as unit power per unit volume (W/m^3 in SI units).

Boundary Condition Types

The available boundary conditions are:

BOUNDARY CONDITION	DESCRIPTION
$\mathbf{n}\cdot(k\nabla T) = q_0 + h(T_{\mathrm{inf}} - T) + C_{\mathrm{const}}(T_{\mathrm{amb}}^4 - T^4)$	Heat flux
$\mathbf{n}\cdot(k\nabla T) = 0$	Insulation or symmetry
$T = T_0$	Prescribed temperature
$T = 0$	Zero temperature
$\mathbf{n}_1\cdot(k_1\nabla T_1) = \frac{k}{d}(T_2 - T_1)$ $\mathbf{n}_2\cdot(k_2\nabla T_2) = \frac{k}{d}(T_1 - T_2)$	Thin thermally resistive layer (pair boundaries only)

HEAT FLUX

$$\mathbf{n}\cdot(k\nabla T) = q_0 + h(T_{\mathrm{inf}} - T) + C_{\mathrm{const}}(T_{\mathrm{amb}}^4 - T^4)$$

COMSOL Multiphysics supplies only one generalized boundary condition for heat flux, but it accounts for general heat flux as well as heat flux from convection and radiation.

The application mode interprets the heat flux q_0 in the direction of the inward normal. It interprets the convection and radiation terms in the direction of the outward normal.

- Specify q_0 to represent a heat flux that enters the domain. For instance, any electric heater is well represented by this condition, and you can omit its geometry. Enter the quantity q_0 as unit power per unit area (W/m^2 using SI units). While this is directly applicable to 3D, unit depth and unit area are assumed in 2D and 1D applications, respectively.
- $h(T_{\mathrm{inf}} - T)$ models convective heat transfer with the surrounding environment, where h is the *heat transfer coefficient* and T_{inf} is the ambient bulk temperature. The value of h depends on the geometry and the ambient flow conditions. For a thorough introduction on how to calculate heat transfer coefficients see Ref. 1.
- $C_{\mathrm{const}}\,(T^4{}_{\mathrm{amb}} - T^4)$ models radiation heat transfer with the surrounding environment. T_{amb} is the temperature of the surrounding radiation environment, which might differ from T_{inf}. C_{const} is the product of the surface emissivity ε and the Stefan-Boltzmann constant $\sigma = 5.669 \cdot 10^{-8}$ W/(m$^2 \cdot$K^4) (with the same unit as the Stefan-Boltzmann constant):

$$C_{\mathrm{const}} = \varepsilon\sigma$$

  The surface emissivity is a material property discussed and tabulated in Ref. 1.

Note: A problem with radiation boundaries is nonlinear, so you must work with absolute (thermodynamical) temperature units. See “Using Units” on page 183 in the *COMSOL Multiphysics User’s Guide*.

INSULATION OR SYMMETRY

$$\mathbf{n} \cdot (k\nabla T) = 0$$

This condition specifies where the domain is well insulated, or it reduces model size by taking advantage of symmetry. Intuitively this equation says that the gradient across the boundary must be zero. For this to be true, the temperature on one side of the boundary must equal the temperature on the other side. Because there is no temperature difference across the boundary, heat cannot transfer across it.

An interesting numerical check for convergence is the numerical evaluation of the above condition along the boundary, something easily accomplished with the postprocessing features of COMSOL Multiphysics. Another check is to plot the temperature field as a contour plot. Ideally the contour lines are perpendicular to any insulated boundary.

AXIAL SYMMETRY

This boundary condition is available only for axisymmetric versions of the heat transfer application models. Use it only on the symmetry axis *r = 0*.

PRESCRIBED TEMPERATURE

$$T = T_0$$

This boundary prescribes the temperature T_0. This condition means that the finite element solution returns a solution in which the above condition is either true or minimally approximated.

PRESCRIBED ZERO TEMPERATURE

$$T = 0$$

This boundary specifies a zero boundary temperature.

CONTINUITY

The default setting for interior boundaries is Continuity, which is a special case of the above Heat source/sink condition. In the absence of sources or sinks, that condition becomes $-\mathbf{n}_1\cdot(\mathbf{q}_1 - \mathbf{q}_2) = 0$. This means that the heat flux in the normal direction is continuous across the boundary. This is the default boundary condition on interior boundaries.

The temperature is naturally continuous on an internal boundary following the continuity of the finite element field. Therefore, the Continuity boundary condition is identical to the condition that applies between any two neighboring elements in the mesh. In fact, as long as you have not selected **Enable interior boundaries** in the **Boundary Settings** dialog box, the interior boundaries are not in any way different from any other mesh element boundaries, where the Continuity condition effectively applies.

The Continuity boundary condition is available on interior boundaries and assembly pairs only.

HEAT FLUX DISCONTINUITY

The Heat flux discontinuity condition is available on interior boundaries and pairs representing borders between parts in an assembly:

$$-\mathbf{n}_1 \cdot \mathbf{q}_1 - \mathbf{n}_2 \cdot \mathbf{q}_2 = q \quad \text{on } \partial\Omega$$

The right-hand side represents a flux discontinuity, or equivalently, a heat source or heat sink depending on sign. The temperature is always continuous due to the continuity of the finite element field. If the right-hand side is zero, this equation specifies continuity also in the normal heat flux.

THIN THERMALLY RESISTIVE LAYER

Use this boundary condition to model a thin layer of thermally resistive material using these equations:

$$\begin{aligned} \mathbf{n}_1 \cdot (k_1 \nabla T_1) &= \frac{k}{d}(T_2 - T_1) \\ \mathbf{n}_2 \cdot (k_2 \nabla T_2) &= \frac{k}{d}(T_1 - T_2) \end{aligned}$$

The layer has the thickness d and the thermal conductivity k. This boundary condition is only available for pairs representing borders between the parts in an assembly.

Boundary Settings

You specify the boundary conditions in the **Boundary Settings** dialog box.

QUANTITY	EDIT FIELD	DESCRIPTION
q_0	q	Inward heat flux
h	h	Convective heat transfer coefficient
T_{inf}	T$_{\text{inf}}$	Ambient bulk temperature
Const	Const	Radiation constant: product of emissivity and Stefan-Boltzmann constant
T_{amb}	T$_{\text{amb}}$	Temperature of the surrounding radiating environment
T_0	T$_0$	Prescribed temperature

Application Mode Variables

The Conduction application mode uses the following variables for domain equations, boundary equations, and postprocessing purposes.

NAME	DOMAIN/ TYPE	DESCRIPTION	EXPRESSION
`T`	S/B	Temperature	T
`gradT, Tx`*`i`*	S/V	Temperature gradient	$\lvert\nabla T\rvert,\ \frac{\partial T}{\partial x_i}$
`flux`	S	Heat flux	$\lvert -\underline{k}\nabla T\rvert$
`nflux_T`	B	Normal heat flux	$\mathbf{n}\cdot(-\underline{k}\nabla T)$
`fluxx`*`i`*	V	Heat flux, x_i component	$\sum_j -k_{ij}\frac{\partial T}{\partial x_j}$
`Dts`	S	Time-scaling coefficient	δ_{ts}
`rho`	S	Density	ρ
`C`	S	Heat capacity	C_p
`k, kx`*`i`*`x`*`j`*	S	Thermal conductivity	k, k_{ij}
`Q`	S	Heat source	Q
`htrans`	S	Transversal convective heat transfer coefficient	h_{trans}
`Text`	S	Transversal external temperature	T_{ext}
`Ctrans`	S	User-defined constant	C_{trans}
`Tambtrans`	S	Transversal ambient temperature	T_{ambtrans}
`T0`	B	Prescribed temperature	T_0
`h`	B	Convective heat transfer coefficient	h
`Tinf`	B	Ambient bulk temperature	T_{inf}
`Const`	B	Radiation constant: product of emissivity and Stefan-Boltzmann constant	Const
`Tamb`	B	Temperature of the surrounding radiating environment	T_{amb}

Note: To form the complete application mode variable names, add a suffix consisting of an underscore and the application mode name (default: `ht`), for example, `flux_ht`. (This does not apply to the dependent variable for the temperature.)

Note: The vector expressions V are not present in the 1D formulation of the Conduction application mode.

The Convection and Conduction Application Mode

In addition to heat transfer by conduction, the Convection and Conduction application mode includes heat transfer by convection. In the convection term for the equation that defines this application mode you can specify the velocity vector as an analytical expression. Alternatively, you can connect it directly to the solution of the equations of motion, for example, through a multiphysics coupling to the Incompressible Navier-Stokes application mode. The Convection and Conduction application mode is suitable for modeling of heat transfer in fluids.

Variables and Space Dimensions

The Convection and Conduction application mode is available in 1D, 2D, 3D, Axial symmetry 1D, and Axial symmetry 2D.

Note: The optional Chemical Engineering Module also contains this Convection and Conduction application mode. In addition to the above-mentioned space dimensions, it features pseudo-2D and pseudo-3D geometry options in which it uses time as a second or third spatial dimension. This feature is useful in situations where convection in the direction of the flow is large. This often applies to reactors and equipment for unit operations.

PDE Formulation

This application mode supports the following nonconservative formulation of the transient PDE for heat transfer by convection and conduction:

$$\delta_{ts}\rho C_p\frac{\partial T}{\partial t} + \nabla\cdot(-k\nabla T) = Q - \rho C_p\mathbf{u}\cdot\nabla T$$

The nonconservative formulation assumes an incompressible fluid, which means that $\nabla\cdot\mathbf{u} = 0$. This ensures that no unphysical source term arises from a flow field where the incompressibility constraint, $\nabla\cdot\mathbf{u} = 0$, is not absolutely fulfilled. The convective term appears on the right-hand side of the equation, which implies that setting the temperature gradient to zero directly expresses the convective boundary condition.

Subdomain Settings

The following table contains the quantities in the equations:

COEFFICIENT	VARIABLE	DESCRIPTION
δ_{ts}	`Dts_T`	Time-scaling coefficient
ρ	`rho_T`	Density
C_p	`C_T`	Heat capacity
k	`k_T`	Thermal conductivity
$\underline{k}$		Thermal conductivity tensor
k_{ij}	`kx`*`i`*`x`*`j`*`_T`	Thermal conductivity tensor, x_ix_j component
Q	`Q_T`	Heat source
u, v, w	`u_T`, `v_T`, `w_T`	Velocity in the x_1, x_2, and x_3 directions

For equations in 2D or 3D, pay special attention to the isotropic thermal conductivity, k. If you select this coefficient, the application mode expands it to the diagonal of the thermal conductivity tensor, that is, k_{ii} equals k.

ARTIFICIAL DIFFUSION

The Convection and Conduction application mode supports artificial diffusion using the following methods:

- Isotropic diffusion
- Streamline diffusion
- Crosswind diffusion

To specify and activate artificial diffusion:

1. Open the **Subdomain Settings** dialog box.
2. Click the **Physics** tab.
3. With at least one subdomain selected, click the **Artificial Diffusion** button.

See “Stabilization Techniques” on page 433 in the *COMSOL Multiphysics Modeling Guide* for more information about artificial diffusion.

Boundary Conditions

The available boundary conditions are:

BOUNDARY CONDITION	DESCRIPTION
$T = T_0$	Temperature
$-\mathbf{n}\cdot(-\underline{k}\nabla T + \rho C_p\mathbf{u}T) = q_0$	Heat flux
$\mathbf{n}\cdot(-k\nabla T + \rho C_p\mathbf{u}T) = 0$	Insulation/symmetry
$\mathbf{n}\cdot(-k\nabla T) = 0$	Convective flux
$\mathbf{n}\cdot(-k\nabla T + \rho C_p\mathbf{u}T) = 0$	Axial symmetry
$-\mathbf{n}_1\cdot\mathbf{q}_1 - \mathbf{n}_2\cdot\mathbf{q}_2 = 0$	Continuity
$-\mathbf{n}_1\cdot\mathbf{q}_1 - \mathbf{n}_2\cdot\mathbf{q}_2 = q$	Heat flux discontinuity
$\mathbf{n}_1\cdot(k_1\nabla T_1) = \frac{k}{d}(T_2 - T_1)$ $\mathbf{n}_2\cdot(k_2\nabla T_2) = \frac{k}{d}(T_1 - T_2)$	Thin thermally resistive layer

The Continuity and Heat flux discontinuity boundary conditions are available on interior boundaries and assembly pairs only. The Thin thermally resistive layer boundary condition is only available on assembly pairs.

The only difference between this application mode and the Conduction application mode is that the heat flux contains the added convection term. In cases where convection across a boundary is much greater than diffusion, use the convective flux boundary condition. It sets the diffusive flux at the boundary to zero, but it allows convective flux to exit the domain.

Note: When working in an axisymmetry application mode, use the axial symmetry boundary condition only on the symmetry axis.

Application Mode Variables

The Convection and Conduction application mode uses the following expressions and coefficients in boundary conditions, equations, and for postprocessing purposes.

NAME	TYPE	DESCRIPTION	EXPRESSION
`T`	S/B	Temperature	T
`grad_T`	S/V	Temperature gradient	$\lvert\nabla T\rvert,\ \frac{\partial T}{\partial x_i}$
`dflux_T`	S	Conductive flux	$\lvert\underline{k}\nabla T\rvert$
`cflux_T`	S	Convective flux	$\lvert\rho C_p T\mathbf{u}\rvert$
`tflux_T`	S	Total heat flux	$\lvert-\underline{k}\nabla T+\rho C_p T\mathbf{u}\rvert$
`ndflux_T`	B	Normal conductive flux	$\mathbf{n}\cdot(-\underline{k}\nabla T)$
`ncflux_T`	B	Normal convective flux	$\rho C_p T\mathbf{n}\cdot\mathbf{u}$
`ntflux_T`	B	Normal total heat flux	$\mathbf{n}\cdot(-\underline{k}\nabla T+\rho C_p T\mathbf{u})$
`dflux_T_x`*`i`*	V	Conductive flux, x_i component	$\sum_j -k_{ij}\frac{\partial T}{\partial x_j}$
`cflux_T_x`*`i`*	V	Convective flux, x_i component	$\rho C_p T u_i$
`tflux_T_x`*`i`*	V	Total heat flux, x_i component	$\sum_j -k_{ij}\frac{\partial T}{\partial x_j}+\rho C_p T u_i$
`cellPe_T`	S	Cell Peclet number	$\left\lvert\frac{\rho C_p \mathbf{u}h}{\underline{k}}\right\rvert$
`Dts_T`	S	Time-scale factor	δ_{ts}
`rho_T`	S	Density	ρ
`C_T`	S	Heat capacity	C_p
`k_T, kx`*`i`*`x`*`j`*`_T`	S	Thermal conductivity	$k,\ k_{ij}$
`Q_T`	S	Heat source	Q
`u_T, v_T, w_T`	S	Velocity of c, x_i component	u_i

NAME	TYPE	DESCRIPTION	EXPRESSION
`Dm_T`	S	Mean diffusion coefficient	$\dfrac{\sum_{i,j} k_{ij}\beta_i\beta_j}{\lvert\beta\rvert}$
`res_T_cc`	S	Equation residual	$\nabla\cdot(-\underline{k}\nabla T+\rho C_p T\mathbf{u})-Q$
`res_sc_T_cc`	S	Shock-capturing residual	$\nabla\cdot(\rho C_p T\mathbf{u})-Q$
`da_T`	S	Total time-scale factor	$\delta_{\mathrm{ts}}\rho C_p$
`q`	B	Inward heat flux	q_0
`T0`	B	Prescribed temperature	T_0

Note: To form the complete application mode variable names, add a suffix consisting of an underscore and the application mode name (default: cc), for example, `tflux_T_cc`. (This does not apply to the dependent variable for the temperature.)

The vector variables, indicated by V in the Type column, are not present in 1D versions of the Convection and Conduction application mode.

Examples of Heat Transfer Models

The following three heat transfer benchmark examples show how to model heat transfer using:

- Steady-state and transient analysis
- Temperature, heat flux, convective cooling, and radiation boundary conditions
- Thermal conductivity as a function of temperature

All examples are taken from a NAFEMS benchmark collection (Ref. 2).

1D Steady-State Heat Transfer with Radiation

The first example shows a 1D steady-state thermal analysis including radiation to a prescribed ambient temperature.

Model Definition

This 1D model has a domain of length 0.1 m. The left end is kept at 1000 K, and at the right end there is radiation to 300 K. For the radiation, the model properties are:

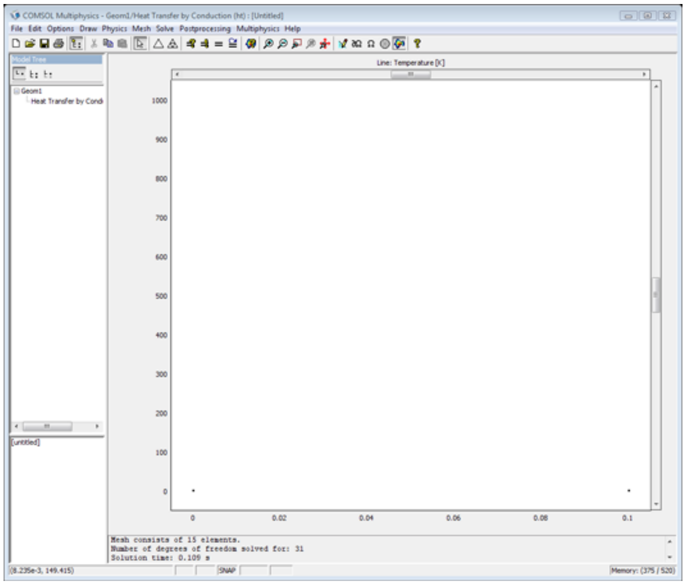
- The emissivity, ε, is 0.98.
- The Stefan-Boltzmann constant, σ, is $5.67 \cdot 10^{-8}$ W/(m·K^4).

In the domain, use the following material property:

- The thermal conductivity is 55.563 W/(m·°C).

Results

The following plot shows the temperature as a function of position:



```
Mesh consists of 15 elements.
Number of degrees of freedom solved for: 31
Solution time: 0.109 s
```

Figure 7-1: Temperature as a function of position.

The benchmark result for the right end is a temperature of 927.0 K. The COMSOL Multiphysics model, using a default mesh with 15 elements, gives a temperature at the end as 926.97 K, which is the exact benchmark value to four significant digits.

Model Library path: `COMSOL_Multiphysics/Heat_Transfer/heat_radiation_1D`

Modeling Using the Graphical User Interface

MODEL NAVIGATOR

1 Go to the **Model Navigator** and select **1D** from the **Space dimension** list.

2 In the list of application modes, open the **COMSOL Multiphysics>Heat Transfer** folder and then the **Conduction** node.

3 Select **Steady-state analysis**.

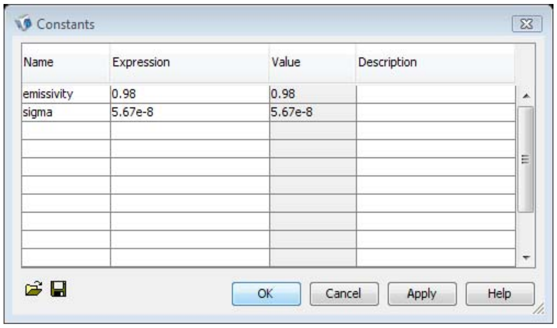
4 Click **OK**.

OPTIONS AND SETTINGS

1 Go to the **Options** menu and choose **Constants**.

2 Enter the following constants for the emissivity and the Stefan-Boltzmann constant:

NAME	EXPRESSION
`emissivity`	`0.98`
`sigma`	`5.67e-8`



Name	Expression	Value	Description
emissivity	0.98	0.98	
sigma	5.67e-8	5.67e-8	

GEOMETRY MODELING

1 Go to the **Draw** menu, point to **Specify Objects** and click **Line**.

2 In the **Line** dialog box, type `0 0.1` in the **x** edit field under **Coordinates**.

3 Click **OK**.

4 Click the **Zoom Extents** button on the Main toolbar.

PHYSICS SETTINGS

Boundary Conditions

1 Go to the **Physics** menu and choose **Boundary Settings**.

2 In the **Boundary Settings** dialog box select boundary 1.

3 In the **Boundary condition** list select **Temperature**.

4 Type `1000` in the **Temperature** edit field.

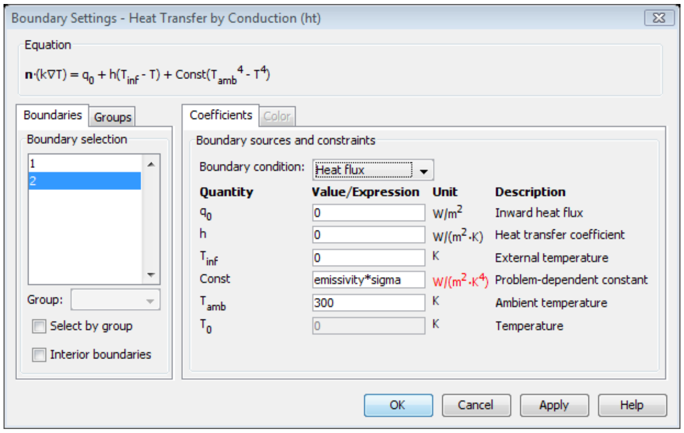
5 Select Boundary 2.

6 From the **Boundary condition** list select **Heat flux**.

7 Type `emissivity*sigma` in the **Problem-dependent constant** edit field.

8 Type `300` in the **Ambient temperature** edit field.

9 Click **OK**.



Boundary Settings - Heat Transfer by Conduction (ht)

Equation

$\mathbf{n}\cdot(k\nabla T) = q_0 + h(T_{inf} - T) + \mathrm{Const}(T_{amb}{}^4 - T^4)$

Boundaries | Groups

Boundary selection: 1, 2 (selected)

Group:

☐ Select by group

☐ Interior boundaries

Coefficients | Color

Boundary sources and constraints

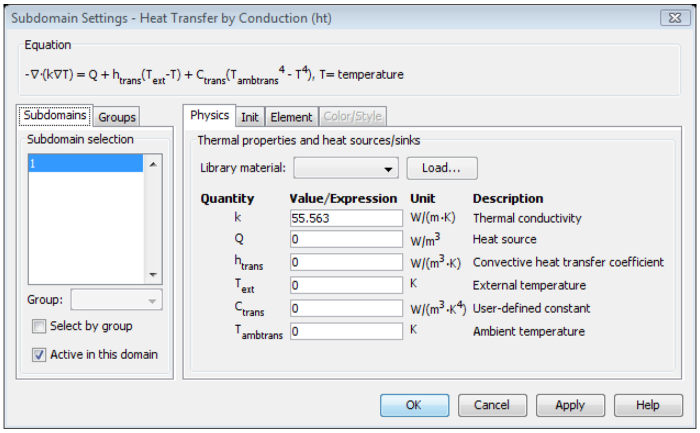
Boundary condition: Heat flux

Quantity	Value/Expression	Unit	Description
q_0	0	W/m^2	Inward heat flux
h	0	$W/(m^2\cdot K)$	Heat transfer coefficient
T_{inf}	0	K	External temperature
Const	emissivity*sigma	$W/(m^2\cdot K^4)$	Problem-dependent constant
T_{amb}	300	K	Ambient temperature
T_0	0	K	Temperature

Subdomain Settings

1 Go to the **Physics** menu and choose **Subdomain Settings**.

2 In the **Subdomain Settings** dialog box enter `55.563` in the **Thermal conductivity** edit field.



Subdomain Settings - Heat Transfer by Conduction (ht)

Equation

$-\nabla\cdot(k\nabla T) = Q + h_{trans}(T_{ext} - T) + C_{trans}(T_{ambtrans}{}^4 - T^4)$, T= temperature

Subdomains | Groups

Subdomain selection: 1 (selected)

Group:

☐ Select by group

☑ Active in this domain

Physics | Init | Element | Color/Style

Thermal properties and heat sources/sinks

Library material: Load...

Quantity	Value/Expression	Unit	Description
k	55.563	$W/(m\cdot K)$	Thermal conductivity
Q	0	W/m^3	Heat source
h_{trans}	0	$W/(m^3\cdot K)$	Convective heat transfer coefficient
T_{ext}	0	K	External temperature
C_{trans}	0	$W/(m^3\cdot K^4)$	User-defined constant
$T_{ambtrans}$	0	K	Ambient temperature

Set the initial value to match the boundary condition. It serves as starting value for the nonlinear solver:

3 Click the **Init** tab.

4 Type 1000 as the initial value in the **Temperature** edit field.

5 Click **OK**.

MESH GENERATION

Initialize the mesh by clicking the **Initialize Mesh** button on the Main toolbar.

COMPUTING THE SOLUTION

Click the **Solve** button on the Main toolbar.

POSTPROCESSING AND VISUALIZATION

Figure 7-1 on page 181 shows the temperature distribution in the domain. Use the zoom tools to focus on the temperature at the right end.

2D Steady-State Heat Transfer with Convection

This example shows a 2D steady-state thermal analysis including convection to a prescribed external (ambient) temperature.

Model Definition

This model domain is 0.6-by-1.0 meters. For the boundary conditions:

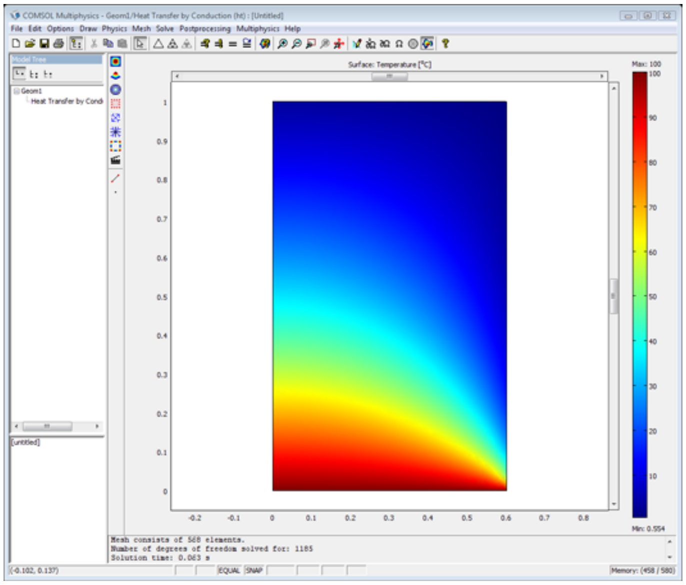
- The left boundary is insulated.
- The lower boundary is kept at 100 °C.
- The upper and right boundaries are convecting to 0 °C with a heat transfer coefficient of 750 W/(m^2·°C).

In the domain use the following material property:

- The thermal conductivity is 52 W/(m·°C).

Results

The following plot shows the temperature as a function of position:



```
Mesh consists of 568 elements.
Number of degrees of freedom solved for: 1185
Solution time: 0.063 s
```

Figure 7-2: Temperature distribution resulting from convection to a prescribed external temperature.

The benchmark result for the target location (x = 0.6 m and y = 0.2 m) is a temperature of 18.25 °C. The COMSOL Multiphysics model, using a default mesh with 556 elements, gives a temperature of 18.28 °C. Successive uniform refinements show temperatures of 18.26 °C and 18.25 °C, converging toward the benchmark result.

Model Library path:

`COMSOL_Multiphysics/Heat_Transfer/heat_convection_2D`

Modeling Using the Graphical User Interface

MODEL NAVIGATOR

1 Go to the **Model Navigator** and select **2D** from the **Space dimension** list.

2 In the list of application modes, open the **COMSOL Multiphysics>Heat Transfer** folder and then the **Conduction** node.

3 Select **Steady-state analysis**.

4 Click **OK**.

GEOMETRY MODELING

1 On the **Draw** menu point to **Specify Objects** and click **Rectangle**.

2 In the **Rectangle** dialog box, find the **Size** area and type `0.6` in the **Width** edit field, then type `1` in the **Height** edit field.

3 Click **OK**.

4 Click the **Zoom Extents** button.

PHYSICS SETTINGS

Boundary Conditions

The default boundary condition is thermal insulation, so you must set boundary conditions for only three of the boundaries.

1 Go to the **Physics** menu and choose **Boundary Settings**.

2 In the **Boundary Settings** dialog box select Boundary 2.

3 In the **Boundary condition** list select **Temperature**.

4 Type `100[degC]` in the **Temperature** edit field to specify a temperature in degrees Celsius (the default unit for temperature in the SI base unit system is kelvin).

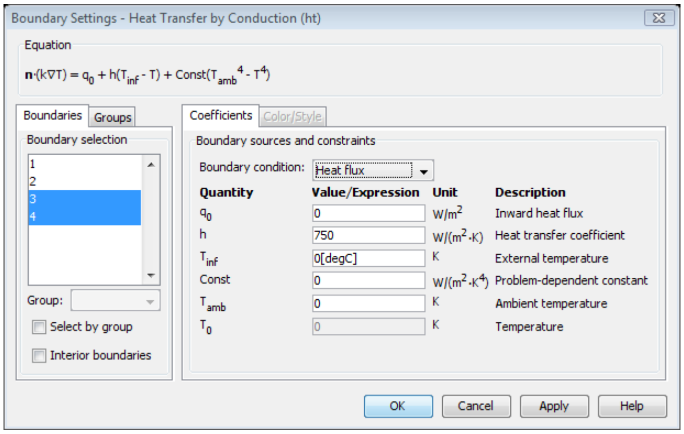
5 Select Boundaries 3 and 4.

6 In the **Boundary condition** list select **Heat flux**.

7 Type `750` in the **Heat transfer coefficien**t edit field.

8 Type `0[degC]` in the **External temperature** edit field.

9 Click **OK**.


Boundary Settings - Heat Transfer by Conduction (ht)

Equation

$\mathbf{n}\cdot(k\nabla T) = q_0 + h(T_{inf} - T) + Const(T_{amb}{}^4 - T^4)$

Boundaries | Groups

Boundary selection: 1, 2, 3 (selected), 4 (selected)

Group:

☐ Select by group

☐ Interior boundaries

Coefficients | Color/Style

Boundary sources and constraints

Boundary condition: Heat flux

Quantity	Value/Expression	Unit	Description
q_0	0	W/m^2	Inward heat flux
h	750	$W/(m^2\cdot K)$	Heat transfer coefficient
T_{inf}	0[degC]	K	External temperature
Const	0	$W/(m^2\cdot K^4)$	Problem-dependent constant
T_{amb}	0	K	Ambient temperature
T_0	0	K	Temperature

Subdomain Settings

1 Go to the **Physics** menu and choose **Subdomain Settings**.

2 In the **Subdomain Settings** dialog box enter `52` in the **Thermal conductivity** edit field.

3 Click **OK**.

MESH GENERATION

Initialize the mesh by clicking the **Initialize Mesh** button on the Main toolbar.

COMPUTING THE SOLUTION

Click the **Solve** button on the Main toolbar.

POSTPROCESSING AND VISUALIZATION

Figure 7-2 on page 185 shows the temperature distribution in the domain. To get a plot showing the numerical value at the reference point, use a cross-section plot:

1 Go to the **Postprocessing** menu and choose **Cross-Section Plot Parameters**.

2 In the **Cross-Section Plot Parameters** dialog box click the **Point** tab.

3 In the **Coordinates** area enter `0.6` in the **x** edit field and `0.2` in the **y** edit field.

4 Select **°C** from the **Unit** list.

5 Click **Apply**.

2D Axisymmetric Transient Heat Transfer

This example shows an axisymmetric transient thermal analysis with a step change to 1000 °C at time 0.

Model Definition

This model domain is 0.3-by-0.4 meters. For the boundary conditions:

- The left boundary is the symmetry axis.
- The other boundaries have a temperature of 1000 °C. The entire domain is at 0 °C at the start, which represents a step change in temperature at the boundaries.

In the domain use the following material properties:

- The density, ρ, is 7850 kg/m^3
- The heat capacity is 460 J/(kg·°C)
- The thermal conductivity is 52 W/(m·°C)

Results

The following plot shows the temperature as a function of position after 190 seconds:

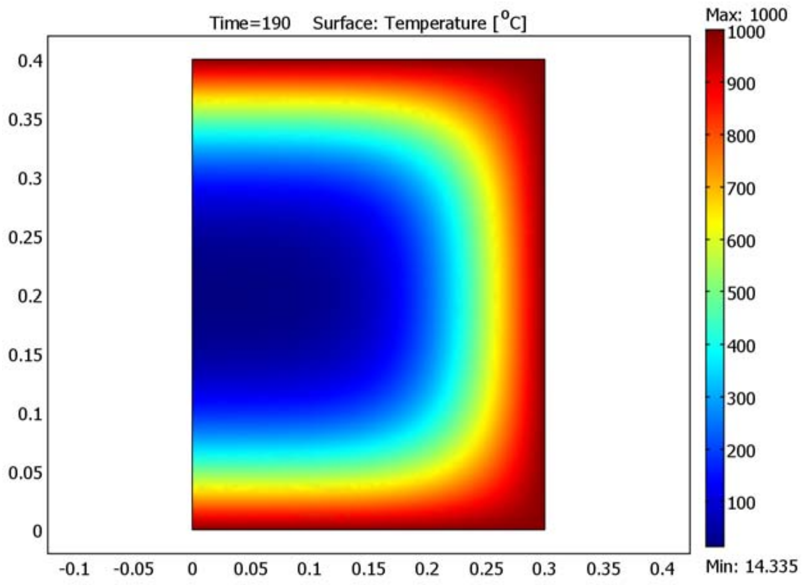


Figure 7-3: Temperature distribution after 190 seconds.

The benchmark result for the target location ($r = 0.1$ m and $z = 0.3$ m) is a temperature of 186.5 °C. The COMSOL Multiphysics model, using a default mesh with about 720 elements, gives a temperature of roughly 186.4 °C.

As an additional postprocessing step, map the axisymmetric solution to 3D using an extrusion coupling variable to show the solution for the entire cylinder (see Figure 7-4).

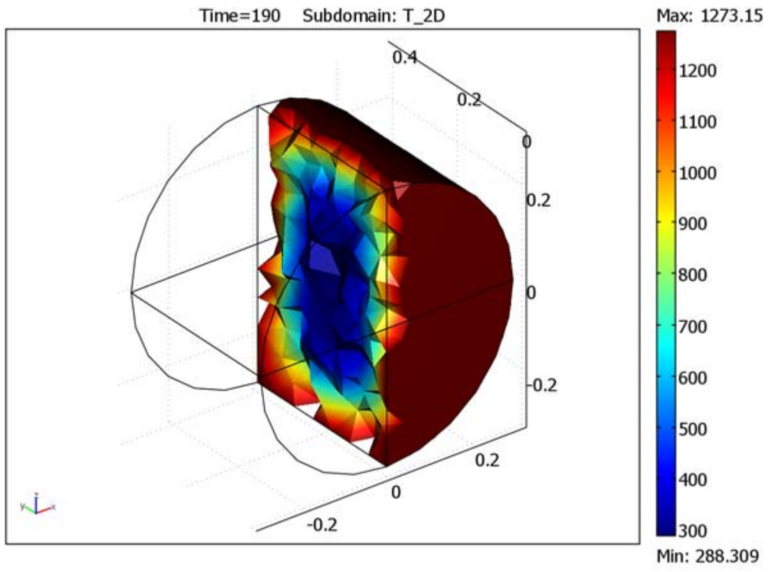


Figure 7-4: Postprocessing of the temperature in the full 3D geometry.

Model Library path:

`COMSOL_Multiphysics/Heat_Transfer/heat_transient_axi`

Modeling Using the Graphical User Interface

MODEL NAVIGATOR

1 Go the **Model Navigator** and select **Axial symmetry (2D)** from the **Space dimension** list.

2 From the list of application modes, select **COMSOL Multiphysics>Heat Transfer> Conduction**.

3 Select **Transient analysis**.

4 Click **OK**.

GEOMETRY MODELING

1 Go to the **Draw** menu, point to **Specify Objects** and click **Rectangle**.

2 In the **Rectangle** dialog box go to the **Size** area and enter `0.3` in the **Width** edit field and `0.4` in the **Height** edit field.

3 Click **OK**.

4 Click the **Zoom Extents** button on the Main toolbar.

PHYSICS SETTINGS

Boundary Conditions

1 Go to the **Physics** menu and choose **Boundary Settings**.

2 In the **Boundary Settings** dialog box select Boundary 1.

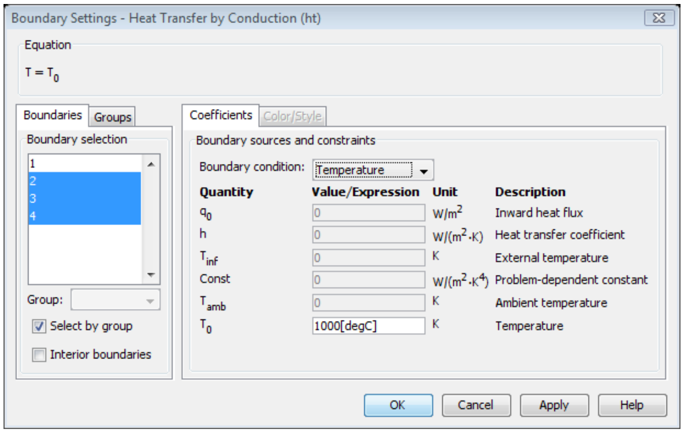
3 In the **Boundary condition** list select **Axial symmetry**.

4 Select the **Select by group** check box and choose Boundaries 2, 3, and 4 by selecting one of them.

5 Select **Temperature** in the **Boundary condition** list.

6 Type `1000[degC]` in the **Temperature** edit field.

7 Click **OK**.,



$T = T_0$

Boundary selection: 1, 2, 3, 4 (2, 3, 4 selected)

Group:

☑ Select by group

☐ Interior boundaries

Boundary sources and constraints

Boundary condition: Temperature

Quantity	Value/Expression	Unit	Description
q_0	0	W/m^2	Inward heat flux
h	0	$W/(m^2 \cdot K)$	Heat transfer coefficient
T_{inf}	0	K	External temperature
Const	0	$W/(m^2 \cdot K^4)$	Problem-dependent constant
T_{amb}	0	K	Ambient temperature
T_0	1000[degC]	K	Temperature

Subdomain Settings

1 Go to the **Physics** menu and choose **Subdomain Settings**.

2 In the **Subdomain Settings** dialog box enter the thermal properties in the domain according to the following table:

PROPERTIES	SUBDOMAIN 1
k (isotropic)	52
ρ	7850
C_p	460

3 Click the **Init** tab.

4 Type `0[degC]` in the **T(t_0)** edit field for the initial temperature.

5 Click **OK**.

MESH GENERATION

Initialize the mesh by clicking the **Initialize Mesh** button on the Main toolbar.

COMPUTING THE SOLUTION

1 Go to the **Solve** menu and choose **Solver Parameters**.

2 In the **Time stepping** area in the **Solver Parameters** dialog box enter `0:10:190` in the **Times** edit field.

3 Click **OK**.

4 Click the **Solve** button on the Main toolbar.

POSTPROCESSING AND VISUALIZATION

Getting the Reference Temperature

Figure 7-3 on page 189 shows the temperature distribution in the domain. To get the numerical value at the reference point, use the **Data Display** dialog box:

1 Choose **Postprocessing>Data Display>Subdomain**.

2 In the **Coordinates** area enter `0.1` in the **r** edit field and `0.3` in the **z** edit field.

3 Click **OK**.

The value of the temperature in the reference point appears in the message log. You can also click in the temperature plot at (0.1, 0.3) to display the temperature at that point in the message log.

Postprocessing in 3D

To postprocess the solution in 3D, first revolve the geometry into a cylinder in a 3D geometry and then map the axisymmetric solution to the cylinder using an extrusion coupling variable:

1 From the **Draw** menu, choose **Revolve**.

2 In the **Revolve** dialog box, leave the default settings and click **OK**. This creates a cylinder in 3D. Note that the axis of revolution in 3D is the y-axis, which means that the plane that you map the radial coordinate r to is the xz-plane.

3 Click the **Geom1** tab at the top of the drawing area to return to the 2D axisymmetric geometry.

4 Choose **Options>Extrusion Coupling Variables>Subdomain Variables**.

5 In the **Subdomain Extrusion Variables** dialog box, select Subdomain 1 and then type `T_2D` in the first row of the **Name** column and `T` is the first row of the **Expression** column. This creates an extrusion coupling variable `T_2D` that represents the temperature (the variable `T`).

6 Click the **General** transformation button. The default source transformation (**x: r** and **y: z**) is correct.

7 Click the **Destination** tab.

8 Select **Geom2** from the **Geometry** list, select **Subdomain** from the **Level** list, and finally select the **1** check box for Subdomain 1 in the Subdomain selection list. The variable `T_2D` is the only extrusion coupling variable and the software selects it automatically.

9 In the **Destination transformation** area, type `sqrt(x^2+z^2)` in the **x** edit field, and leave the value **y** in the **y** edit field. This transforms r and z in the axisymmetric geometry to $\sqrt{x^2+z^2}$ and y, respectively, in the 3D geometry.

10 Click **OK**.

11 From the **Solve** menu, choose **Update Model** to map the solution to the 3D geometry.

12 From the **Postprocessing** menu, choose **Plot Parameters**.

13 To plot the temperature in the cylinder as a subdomain plot, clear the **Slice** check box and select the **Subdomain** check box in the **Plot type** area on the **General** page.

14 Select the **Element selection** check box and type `x>0` in the **Logical expression** edit field to "look inside" by removing all elements with a negative x coordinate.

15 Click the **Subdomain** tab, and then type `T_2D` in the **Expression** edit field.

16 Click **OK** to generate Figure 7-4 on page 190.

References

1. Frank P. Incropera and David P. DeWitt, *Fundamentals of Heat and Mass Transfer*, 4th edition, John Wiley & Sons, New York, 1996.

2. A.D. Cameron, J. A. Casey, and G.B. Simpson: *NAFEMS Benchmark Tests for Thermal Analysis (Summary)*, NAFEMS Ltd., Glasgow, 1986.

8 Structural Mechanics

This chapter explains how to use the Structural Mechanics application modes to simulate and analyze applications involving solid mechanics. It begins with a brief theoretical backgrounder on structural mechanics, after which subsequent sections give details of the application modes.

The Structural Mechanics Application Modes

COMSOL Multiphysics includes four application modes for stress analysis and general structural mechanics simulation:

- The Solid, Stress-Strain application mode (for 3D geometries)
- The Plane Stress application mode (for 2D geometries)
- The Plane Strain application mode (for 2D geometries)
- The Axial Symmetry Stress-Strain application mode (for 2D axisymmetric geometries)

The last three cases are 2D simplifications of the full 3D equations, simplifications that are valid under certain assumptions.

Note: The optional Structural Mechanics Module contains application modes and models that allow for extended, specialized analyses of structural and solid mechanics problems.

Theory Background

Strain-Displacement Relationship

It is possible to completely describe the strain conditions at a point with the deformation components—(u, v, w) in 3D—and their derivatives. You can express the shear strain in a tensor form, ε_{xy}, ε_{yz}, ε_{xz}, or in an engineering form, γ_{xy}, γ_{yz}, γ_{xz}. Following the small-displacement assumption, the normal strain components and the shear strain components are given from the deformation as follows:

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_{xy} = \frac{\gamma_{xy}}{2} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \varepsilon_{yz} = \frac{\gamma_{yz}}{2} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \varepsilon_{xz} = \frac{\gamma_{xz}}{2} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

The symmetric strain tensor ε consists of both normal and shear strain components:

$$\varepsilon = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_z \end{bmatrix}$$

Stress-Strain Relationship

The stress in a material is described by the symmetric stress tensor

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \qquad \tau_{xy} = \tau_{yx} \qquad \tau_{xz} = \tau_{zx} \qquad \tau_{yz} = \tau_{zy}$$

consisting of three normal stresses (σ_x, σ_y, σ_z) and six, or if symmetry is used, three shear stresses (τ_{xy}, τ_{yz}, τ_{xz}). The stress-strain relationship for linear conditions reads:

$$\sigma = D\varepsilon$$

where D is the 6×6 elasticity matrix, and the stress and strain components are described in vector form with the six stress and strain components in column vectors defined as

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix} \qquad \varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix}$$

Note: The following descriptions use the compact notation σ and ε, meaning either the stress/strain vector or tensor depending on the context.

The elasticity matrix D and the more basic matrix D^{-1} (the inverse of D, also known as the flexibility or compliance matrix) are defined differently for isotropic, orthotropic, and anisotropic material. For isotropic materials, the D^{-1} matrix looks like

$$D^{-1} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}$$

where E is the modulus of elasticity (also known as *Young's modulus*), and ν is *Poisson's ratio*, which defines contraction in the perpendicular direction. Inverting D^{-1}

$$D = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Implementation

COMSOL Multiphysics bases its implementation of the structural mechanics application modes on the equilibrium equations expressed in the global stress components.

EQUILIBRIUM EQUATION

The equilibrium equations expressed in the stresses for 3D are

$$-\frac{\partial\sigma_x}{\partial x}-\frac{\partial\tau_{xy}}{\partial y}-\frac{\partial\tau_{xz}}{\partial z}=F_x$$
$$-\frac{\partial\tau_{xy}}{\partial x}-\frac{\partial\sigma_y}{\partial y}-\frac{\partial\tau_{yz}}{\partial z}=F_y$$
$$-\frac{\partial\tau_{xz}}{\partial x}-\frac{\partial\tau_{yz}}{\partial y}-\frac{\partial\sigma_z}{\partial z}=F_z$$

where **F** denotes the volume forces (body forces).

Using compact notation, you can write this relationship as

$$-\nabla\cdot\sigma = \mathbf{F}$$

where σ is the stress tensor. Substituting the stress-strain and strain-displacement relationships in the above equation results in Navier's equation expressed in the displacement.

SETTING UP EQUATIONS FOR THE DIFFERENT ANALYSES

All the application modes for structural mechanics support the following analysis types:

- Static analysis

- Eigenfrequency analysis
- Transient analysis

The equations to solve and the solvers to use vary with the analysis types and implementations. You specify them with the Analysis Type and Implementation application mode properties.

Static Analysis

These analysis types all use the same equation but invoke a different solver. The following discussion uses static analyses to represent all these types of analysis because they use the same equations.

Substitution of the stress-strain relationship and the strain-displacement relationship into the static equilibrium equation produces Navier's equation of equilibrium expressed in the displacements.

For static conditions including temperature, Navier's equation reads

$$-\nabla\cdot(c\nabla\mathbf{u}) = \mathbf{F}$$

Details about the corresponding coefficients appear in the application mode implementation sections.

Eigenfrequency Analysis

The difference between this analysis type and the static analysis is that it adds the mass.

If you wish to solve for the eigenfrequencies, use the following formulation:

$$-\lambda d_a\mathbf{u}-\nabla\cdot c_{\mathrm{sta}}\nabla\mathbf{u} = 0$$

The eigenfrequency *f* relates to the eigenvalue λ as

$$f = \frac{\sqrt{\lambda}}{2\pi}$$

The density appears in the d_a coefficient. In COMSOL Multiphysics you can work with either eigenvalues or eigenfrequencies.

Transient Analysis

A transient problem requires the introduction of Newton's Second Law.

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2}-\nabla\cdot c\nabla\mathbf{u} = \mathbf{F}$$

COMSOL Multiphysics uses Rayleigh damping to model viscous damping by specifying two damping coefficients. To see how the software does this, consider a system with a single degree of freedom. You can describe the motion for a system with viscous damping and with a single degree of freedom as

$$m\frac{d^2u}{dt^2}+\xi\frac{du}{dt}+ku = f(t) \tag{8-1}$$

The Rayleigh damping model expresses the damping parameter ξ in terms of the mass m and the stiffness k as

$$\xi = \alpha_{dM}m+\beta_{dK}k$$

You can reduce the 2nd-order system in Equation 8-1 to a 1st-order system in time by introducing new variables $\mathbf{v} = (u\_t, v\_t, w\_t)^t$

$$\mathbf{v} = \frac{\partial}{\partial t}\mathbf{u}$$

The expanded system look like the following, with the Rayleigh damping modeled in an equivalent way to the single-degree-of-freedom system shown above:

$$\begin{bmatrix} I & 0 \\ 0 & \rho \end{bmatrix}\frac{\partial}{\partial t}\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} - \nabla\cdot\left[\begin{bmatrix} 0 & 0 \\ c_{\mathrm{sta}} & c_{\mathrm{sta}}\beta_{dK} \end{bmatrix}\nabla\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}\right] + \begin{bmatrix} 0 & -I \\ 0 & \alpha_{dM}\rho \end{bmatrix}\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{K} \end{bmatrix}$$

The difference between this system and the scalar case with just one degree of freedom is that the damping parameters are defined locally, meaning that they can vary in space and between subdomains.

Application Mode Descriptions

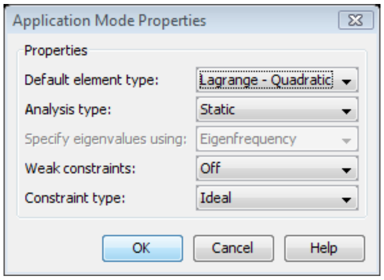
This section describes how to define a structural mechanics model using the application modes. It consists of the following sections:

- Application mode properties
- Material properties
- Constraints
- Loads

The dialog boxes and options are similar for all of the application modes. However, the number of independent variables (spatial coordinates) and dependent variables (displacements) as well as their names vary depending on the application mode. For details on additional application-specific properties, see the following sections, which describe the application modes.

Application Mode Properties

To specify application mode properties, choose **Properties** from the **Physics** menu to open the **Application Mode Properties** dialog box.



Application Mode Properties

Properties

Default element type: Lagrange - Quadratic

Analysis type: Static

Specify eigenvalues using: Eigenfrequency

Weak constraints: Off

Constraint type: Ideal

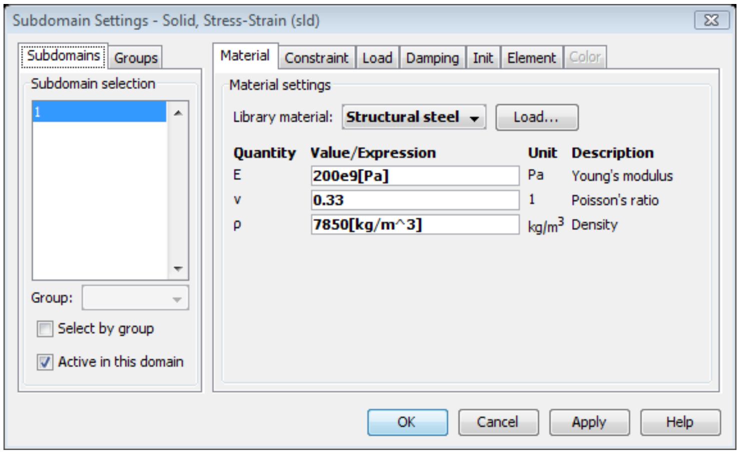
In this dialog box you can specify different global settings for the model:

- Make a selection from the **Default element type** list. This selection becomes the default on all subdomains. Available elements are:
  - **Lagrange - Linear**, 1st-order Lagrange elements
  - **Lagrange - Quadratic**, 2nd-order Lagrange elements
  - **Lagrange - Cubic**, 3rd-order Lagrange elements
  - **Lagrange - Quartic**, 4th-order Lagrange elements
  - **Lagrange - Quintic**, 5th-order Lagrange elements
- Make a selection from the **Analysis type** list; the default is static analysis. Your choice affects the equations and which solver COMSOL Multiphysics uses if you previously selected the **Auto select solver** check box in the **Solver Parameters** dialog box.
- Indicate how you prefer to specify eigenvalues—using eigenvalues themselves or eigenfrequencies—by selecting from the **Specify eigenvalues using** list. This property is applicable only to eigenfrequency analyses.
- Control whether or not weak constraints should be available in the **Weak constraints** list by selecting **On** or **Off**.

  Use weak constraints for accurate computation of reaction forces, solving for the reaction force, which is equal to the Lagrange multiplier that the weak constrain introduces as an additional dependent variable. When weak constraints are enabled, all constraints are weak constraints by default, but it is possible to change this setting for individual domains.
- Choose the type of constraint. Select **Ideal** or **Non-ideal** from the **Constraint type** list. See "Ideal vs. Non-Ideal Constraints" on page 301 for more information about these constrain types.

Material Properties

You define material properties in a model by going to the **Subdomain Settings** dialog box and then the **Material** page.



Subdomain Settings - Solid, Stress-Strain (sld)

Library material: Structural steel

Quantity	Value/Expression	Unit	Description
E	200e9[Pa]	Pa	Young's modulus
ν	0.33	1	Poisson's ratio
ρ	7850[kg/m^3]	kg/m^3	Density

The material properties for the structural mechanics application modes appear in the table below.

PARAMETER	VARIABLE	DESCRIPTION	COMMENT
E	`E`	Young's modulus	
ν	`nu`	Poisson's ratio	
ρ	`rho`	Density	
thickness	`thickness`	Thickness of the structure	Plane stress and plane strain only

Young's modulus Defines the material's modulus of elasticity, E. For isotropic materials it is the spring stiffness in Hooke's law, shown in 1D form as

$$\sigma = E\varepsilon$$

where σ is the stress and ε is the strain.

Poisson's ratio Denoted by ν, it defines the normal strain in the perpendicular direction generated from a normal strain in the other direction.

$$\varepsilon_{\perp} = -\upsilon\varepsilon_{||}$$

Density This property, ρ, specifies the material's density.

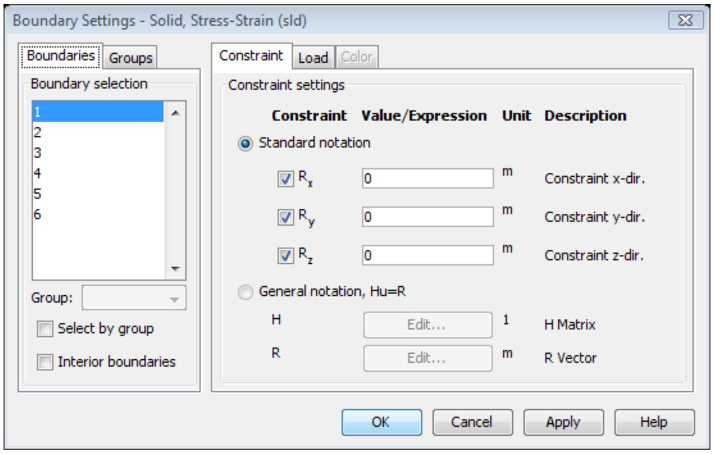
The COMSOL Multiphysics materials library contains values for many of these properties for common materials such as steel, copper, or aluminum. See "Using the Materials/Coefficients Library" on page 223 in the *COMSOL Multiphysics User's Guide*.

Constraints

A constraint specifies the displacement of certain parts of a structure. You can define constraints on all domain levels:

- Points, edges, boundaries (faces), and subdomains in 3D
- Points, boundaries, and subdomains in 2D.

To specify them, go to the **Constraint** page in the **Subdomain Settings**, **Boundary Settings**, **Edge Settings**, or **Point Settings** dialog boxes. The following image shows the **Boundary Settings** dialog box for the Solid, Stress-Strain application mode, but this page looks the same on all domain levels in all structural mechanics application modes.



Boundary Settings - Solid, Stress-Strain (sld)

Boundaries | Groups

Boundary selection: 1 (selected), 2, 3, 4, 5, 6

Group:

☐ Select by group

☐ Interior boundaries

Constraint | Load | Color

Constraint settings

Constraint	Value/Expression	Unit	Description
● Standard notation			
☑ R_x	0	m	Constraint x-dir.
☑ R_y	0	m	Constraint y-dir.
☑ R_z	0	m	Constraint z-dir.
○ General notation, Hu=R			
H	Edit...	1	H Matrix
R	Edit...	m	R Vector

OK | Cancel | Apply | Help

You can describe a constraint using standard or general notation, an option you control using the **Standard notation** and the **General notation, Hu=R** buttons.

STANDARD NOTATION FOR CONSTRAINTS

Using standard notation, you constrain each displacement direction independently:

1 Select the check boxes in front of **Rx**, **Ry**, and **Rz** to activate the constraint.

2 Enter the value or expression for the displacement in the edit fields. The default value is 0.

GENERAL NOTATION FOR CONSTRAINTS

Using general notation (illustrated here for the 2D case), you define the H matrix and R vector in the relationship

$$H\begin{bmatrix} u \\ v \end{bmatrix} = R$$

to specify constraints as any linear combination of displacements components.

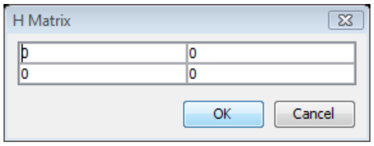
Enter the H matrix and R vector in special matrix dialog boxes by clicking the **Edit** buttons.

For example, to set the condition $u = v$, use the settings

$$H = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \qquad R = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which allow the domain to move only diagonally in the x-y plane.

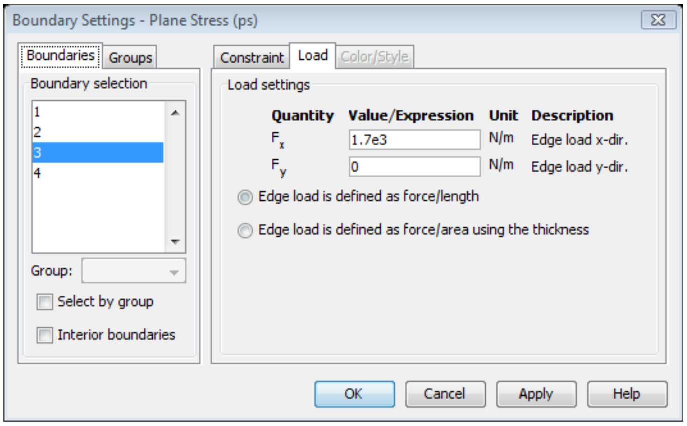
The **H Matrix** dialog box for the above example is



0	0
0	0

Loads

A load is a general name for forces applied to a structure. You can specify loads on all domain types. To specify them, go to the **Load** page in the **Subdomain Settings**, **Boundary Settings**, **Edge Settings**, or **Point Settings** dialog boxes. The image below shows the **Boundary Settings** dialog box for the Plane Stress application mode, but this page looks similar on all domain levels in all structural mechanics application modes.


Boundary Settings - Plane Stress (ps)

Boundary selection: 1, 2, 3 (selected), 4

Load settings

Quantity	Value/Expression	Unit	Description
F_x	1.7e3	N/m	Edge load x-dir.
F_y	0	N/m	Edge load y-dir.

Edge load is defined as force/length

Edge load is defined as force/area using the thickness

When dealing with plane stress and plane strain, you can optionally specify a load in different ways using the thickness. The following table illustrates how to define loads on different domains in different application modes; the SI unit appears in parentheses.

APPLICATION MODE	POINT	EDGE	BOUNDARY	SUBDOMAIN
Plane Stress, Plane Strain	force (N)		force/area (N/m^2) or force/length (N/m)	force/volume (N/m^3) or force/area (N/m^2)
Axisymmetry, Stress-Strain	total force along the circumference (N)		force/area (N/m^2)	force/volume (N/m^3)
Solid, Stress-Strain	force (N)	force/length (N/m)	force/area (N/m^2)	force/volume (N/m^3)

Damping

To specify damping, click the **Damping** tab in the **Subdomain Settings** dialog box. You can choose between *Rayleigh damping* and no damping. For specifying the Rayleigh damping, there are two damping parameters:

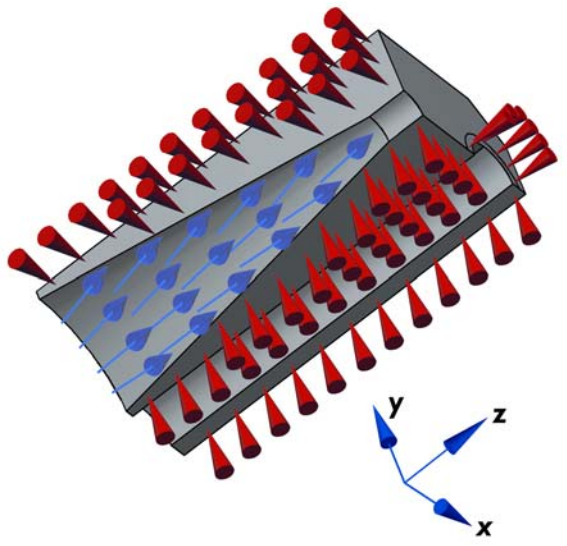
Mass damping parameter Defines the Rayleigh damping model's mass damping, α_{dM}.

Stiffness damping parameter Defines the Rayleigh damping model's stiffness damping, β_{dK}.

α_{dM}	`alphadM`	Mass damping parameter	
β_{dK}	`betadK`	Stiffness damping parameter	

The Solid, Stress-Strain Application Mode

This section explains how to use the Solid, Stress-Strain application mode for stress and strain analysis of 3D solids.



A 3D solid with applied loads and constraints in a stress analysis.

Variables and Space Dimensions

The degrees of freedom (dependent variables) are the global displacements u, v, and w in the global x, y, and z directions.

PDE Formulation

COMSOL Multiphysics formulates this application mode using the equilibrium equations described in general terms in the section "Implementation" on page 199.

APPLICATION MODE PARAMETERS

The parameters that define the loads, material, and constraints are reviewed in the section "Application Mode Descriptions" on page 202.

Application Mode Variables

A large number of variables, listed below, are available for use in expressions and for postprocessing. The table uses an index convention, where a single index on `ui` (u_i) means that i runs over the global displacements (u, v, w), whereas a single index on other names like `si` (σ_i) means that i runs over the global space variables (x, y, z), and a double index `sij` (τ_{ij}) means that ij runs over the combinations (xy, yz, xz) of the space variables.

In addition to the variables in the table, almost all application mode parameters are available as variables. Some are different for some analyses; see the Analysis column, which uses the following abbreviations:

ANALYSIS	ABBREVIATION
Static	S
Time dependent	T

TABLE 8-1: SOLID, STRESS-STRAIN APPLICATION MODE VARIABLES

NAME	SYMBOL	ANALYSIS	DOMAIN	DESCRIPTION	EXPRESSION
`ui`	u_i	All	All	x_i displacement	u_i
`ui_t`	u_{it}	T	All	x_i velocity	u_{it}
`disp`	disp	All	All	Total displacement	$\sqrt{\sum_i (\mathrm{real}(u_i))^2}$
`ex`	ε_x	All	S	ε_x normal strain global system	$\frac{\partial u_i}{\partial x_i}$
`eij`	ε_{ij}	All	S	ε_{ij} shear strain global coord. system	$\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$
`ei_t`	ε_{it}	T	S	ε_{it} normal velocity strain global system	$\frac{\partial u_{it}}{\partial x_i}$

TABLE 8-1: SOLID, STRESS-STRAIN APPLICATION MODE VARIABLES

NAME	SYMBOL	ANALYSIS	DOMAIN	DESCRIPTION	EXPRESSION
`eij_t`	ε_{ijt}	T	S	ε_{ijt} shear velocity strain global coord. system	$\frac{1}{2}\left(\frac{\partial u_{it}}{\partial x_j}+\frac{\partial u_{jt}}{\partial x_i}\right)$
`en`	ε_n	All	S	nth principal strain, n = 1, 2, 3	
`eni`	ε_{ni}	All	S	nth principal strain component in the x_i direction	
`si`	σ_i	All	S	σ_i normal stress global coord. system	$D\varepsilon$
`sij`	τ_{ij}	All	S	τ_{ij} shear stress global coord. system	$D\varepsilon$
`si_t`	σ_{it}	T	S	σ_{it} time derivative of normal stress global coord. system	$D\varepsilon_t$
`sij_t`	τ_{ijt}	T	S	τ_{ijt} time derivative of shear stress global coord. system	$D\varepsilon_t$
`sn`	σ_n	All	S	nth principal stress, n = 1,2,3	
`sni`	σ_{ni}	All	S	nth principal stress, n = 1,2,3 component in the x_i direction	
`tresca`	σ_{tresca}	All	S	Tresca stress	$\max(\max(\lvert\sigma_1-\sigma_2\rvert, \lvert\sigma_2-\sigma_3\rvert), \lvert\sigma_1-\sigma_3\rvert)$
`mises`	σ_{mises}	All	S	von Mises stress	

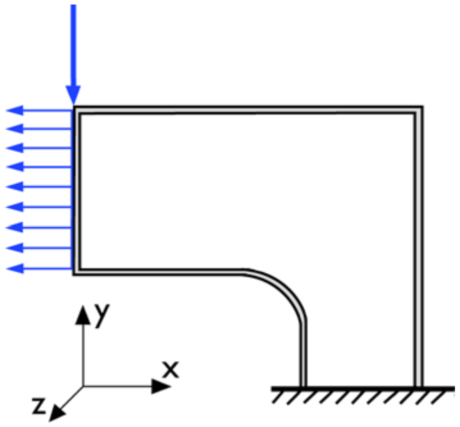
TABLE 8-1: SOLID, STRESS-STRAIN APPLICATION MODE VARIABLES

NAME	SYMBOL	ANALYSIS	DOMAIN	DESCRIPTION	EXPRESSION
`Tai`	Ta_i	All	B	Surface traction (force/area) in x_i direction	$\begin{bmatrix}\mathrm{Ta}_x\\ \mathrm{Ta}_y\\ \mathrm{Ta}_z\end{bmatrix} = \begin{bmatrix}\sigma_x & \tau_{xy} & \tau_{xz}\\ \tau_{xy} & \sigma_y & \tau_{yz}\\ \tau_{xz} & \tau_{yz} & \sigma_z\end{bmatrix}\begin{bmatrix}n_x\\ n_y\\ n_z\end{bmatrix}$
`Fig`	F_{ig}	All	All	Body load, face load, edge load, point load, in global x_i direction	$\begin{bmatrix}F_{xg}\\ F_{yg}\\ F_{zg}\end{bmatrix} = \begin{bmatrix}F_x\\ F_y\\ F_z\end{bmatrix}$

Note: To form the complete application mode variable names, add a suffix consisting of an underscore and the application mode name (default: sld), for example, `mises_sld`. (This does not apply to the dependent variables for the displacements.)

The Plane Stress Application Mode

Use the Structural Mechanics Plane Stress application mode to analyze thin in-plane loaded plates. This application mode solves for the global displacements (u, v) in the x and y directions. In a state of plane stress the σ_z, τ_{yz}, and τ_{xz} components of the stress tensor are assumed to be zero.



Loads and constraints in a plane stress analysis.

Loads in the x and y directions are allowed. The mode assumes that the loads are constant throughout the thickness of the material but that thickness can vary in the x and y directions. The plane stress condition prevails in a thin flat plate in the xy-plane loaded only in its own plane and without any z-direction restraint.

Material

An additional material parameter for plane stress is the plate's thickness.

PARAMETER	VARIABLE	DESCRIPTION	MATERIAL MODEL
th	`thickness`	The plate's thickness	All

PDE Formulation

COMSOL Multiphysics formulates the application mode using the equilibrium equations described in general terms in the section "Implementation" on page 199.

APPLICATION MODE PARAMETERS

The application mode parameters defining the loads, material and constraints are explained in the section "Application Mode Descriptions" on page 202.

Application Mode Variables

A large number of variables are available for use in expressions and for postprocessing. Table 8-2 uses an index convention where a single index on `ui` (u_i) means that i runs over the global displacements (u, v), whereas a single index on other names like `si` (σ_i) means that `i` runs over the global space coordinates (x, y). A single index on x_i means that i runs over the global space coordinates (x, y).

In addition to the variables listed below, almost all application mode parameters are available as variables. Some vary for different analyses, as seen in the Analysis column.

ANALYSIS	ABBREVIATION
Static	S
Time dependent	T

TABLE 8-2: PLANE STRESS APPLICATION MODE VARIABLES

NAME	SYMBOL	ANALYSIS	DOMAIN	DESCRIPTION	EXPRESSION
`u`$_i$	u_i	All	All	x_i displacement	u_i
`ui_t`	u_{it}	T	All	x_i velocity	u_{it}
`disp`	disp	All	All	Total displacement	$\sqrt{\sum_i (\mathrm{real}(u_i))^2}$
`ei`	ε_i	All	S	ε_i normal strain global system	$\frac{\partial u_i}{\partial x_i}$
`ez`	ε_z	All	S	ε_z normal strain out of the x-y plane	$-\frac{\left(\sum_{k=1,2,4} D_{3k}(\varepsilon_k)\right)}{D_{33}}$
`exy`	ε_{xy}	All	S	ε_{xy} shear strain global coord. system	$\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$

TABLE 8-2: PLANE STRESS APPLICATION MODE VARIABLES

NAME	SYMBOL	ANALYSIS	DOMAIN	DESCRIPTION	EXPRESSION
`ei_t`	ε_{it}	T	S	ε_{it} velocity normal strain global coord. system	$\frac{\partial u_{it}}{\partial x_i}$
`ez_t`	ε_{zt}	T	S	ε_{zt} velocity normal strain out of the x-y plane	$-\frac{\sum_{k=1,2,4} D_{3k}\varepsilon_{kt}}{D_{33}}$
`exy_t`	ε_{xyt}	All	S	ε_{xyt} velocity shear strain global coord. system	$\frac{1}{2}\left(\frac{\partial u_t}{\partial y}+\frac{\partial v_t}{\partial x}\right)$
`en`	ε_n	All	S	n:th principal strain, n = 1, 2, 3	
`eni`	ε_{ni}	All	S	n:th principal strain component in the x_i direction	
`si`	σ_i	All	S	σ_i normal stress global coord. system	$\sum_k (D_{ik}(\varepsilon_k))$
`sxy`	τ_{xy}	All	S	τ_{xy} shear stress global coord. system	$\sum_k (D_{4k}\varepsilon_k)$
`si_t`	σ_{it}	T	S	σ_{it} time derivative of normal stress global coord. system	$\sum_k D_{ik}\varepsilon_{kt}$
`sxy_t`	τ_{xyt}	T	S	τ_{xyt} time derivative of shear stress global coord. system	$\sum_k D_{4k}\varepsilon_{kt}$

TABLE 8-2: PLANE STRESS APPLICATION MODE VARIABLES

NAME	SYMBOL	ANALYSIS	DOMAIN	DESCRIPTION	EXPRESSION
`sn`	σ_n	All	S	n st/nd principal stress, n = 1, 2, 3	
`sn`*`i`*	σ_{ni}	All	S	nth principal stress, n = 1, 2, 3 component in the x_i direction	
`tresca`	σ_{tresca}	All	S	Tresca stress	$\mathrm{max}(\mathrm{max}(\lvert\sigma_1-\sigma_2\rvert, \lvert\sigma_2-\sigma_3\rvert)), \lvert\sigma_1-\sigma_3\rvert))$
`mises`	σ_{mises}	All	S	von Mises stress	
`Ta`*`i`*	Ta_i	All	B	Surface traction (force/area) in x_i direction	$\begin{bmatrix}\mathrm{Ta}_x\\ \mathrm{Ta}_y\end{bmatrix} = \begin{bmatrix}\sigma_x & \tau_{xy}\\ \tau_{xy} & \sigma_y\end{bmatrix}\begin{bmatrix}n_x\\ n_y\end{bmatrix}$
`F`*`i`*`g`	F_{ig}	All	S	Body load, in global x_i direction	If force/area $\begin{bmatrix}F_{xg}\\ F_{yg}\end{bmatrix} = \begin{bmatrix}F_x\\ F_y\end{bmatrix}$ If force/volume $\begin{bmatrix}F_{xg}\\ F_{yg}\end{bmatrix} = \mathrm{th}\begin{bmatrix}F_x\\ F_y\end{bmatrix}$

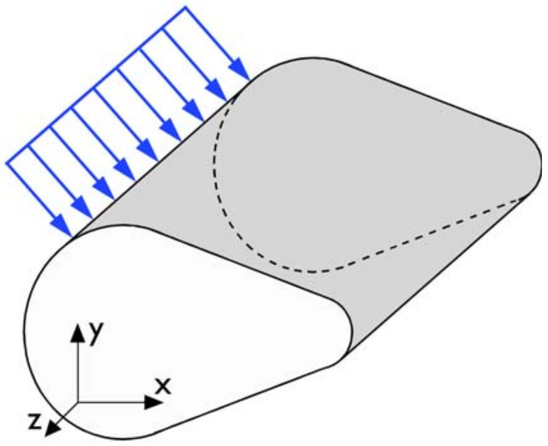
TABLE 8-2: PLANE STRESS APPLICATION MODE VARIABLES

NAME	SYMBOL	ANALYSIS	DOMAIN	DESCRIPTION	EXPRESSION
`Fig`	F_{ig}	All	B	Edge load in global x_i direction	If force/length $\begin{bmatrix} F_{xg} \\ F_{yg} \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \end{bmatrix}$ If force/area $\begin{bmatrix} F_{xg} \\ F_{yg} \end{bmatrix} = \mathrm{th} \begin{bmatrix} F_x \\ F_y \end{bmatrix}$
`Fig`	F_{ig}	All	P	Point load in global x_i direction	$\begin{bmatrix} F_{xg} \\ F_{yg} \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \end{bmatrix}$

Note: To form the complete application mode variable names, add a suffix consisting of an underscore and the application mode name (default: ps), for example, `mises_ps`. (This does not apply to the dependent variables for the displacements.)

The Plane Strain Application Mode

The Plane Strain application mode solves for the global displacements (u, v) in the x and y directions. In a state of plane strain, the ε_z, ε_{yz}, and ε_{xz} components of the strain tensor are assumed to be zero.



Loads in a plane strain analysis.

Loads in the x and y directions are allowed. The loads are assumed to be constant throughout the thickness of the material, but that thickness can vary in the x and y directions. The plane strain condition prevails in geometries that extend much farther in the z direction than in the x and y directions, or when the z-displacement is in some way restricted. The 2D geometry in a plane strain model represents a cross section that cuts a very long or infinite depth such that you can ignore any end effects. An example is a long tunnel along the z-axis where it is sufficient to study a unit-depth slice in the xy-plane. A plane strain model is sometimes also referred to as a ***unit-depth model***.

Material Properties

An additional material property for plane strain analysis is the geometry's thickness.

PARAMETER	VARIABLE	DESCRIPTION	MATERIAL MODEL
th	`thickness`	The geometry's thickness	All

PDE Formulation

COMSOL Multiphysics formulates the application mode using the equilibrium equations described in general terms in the section "Implementation" on page 199.

APPLICATION MODE PARAMETERS

The application mode parameters defining the loads, material, and constraints are explained in the section "Application Mode Descriptions" on page 202.

Application Mode Variables

A large number of variables are available for use in expressions and for postprocessing. In addition to the variables listed below, almost all application mode parameters are available as variables. Some vary for different analyses as seen in the Analysis column.

ANALYSIS	ABBREVIATION
Static	S
Time dependent	T

Table 8-3 uses an index convention where a single index on `ui` (u_i) means that i runs over the global displacements (u, v), whereas a single index on other names like `si` (σ_i) means that `i` runs over the global spatial coordinates (x, y). A single index on x_i means that i runs over the global spatial coordinates (x, y).

TABLE 8-3: PLANE STRAIN APPLICATION MODE VARIABLES

NAME	SYMBOL	ANALYSIS	DOMAIN	DESCRIPTION	EXPRESSION
`ui`	u_i	All	All	x_i displacement	u_i
`ui_t`	u_{it}	T	All	x_i velocity	u_{it}
`disp`	disp	All	All	Total displacement	$\sqrt{\sum_i (\mathrm{real}(u_i))^2}$
`ei`	ε_i	All	S	ε_i normal strain global system	$\frac{\partial u_i}{\partial x_i}$
`exy`	ε_{xy}	All	S	ε_{xy} shear strain global coord. system	$\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$

TABLE 8-3: PLANE STRAIN APPLICATION MODE VARIABLES

NAME	SYMBOL	ANALYSIS	DOMAIN	DESCRIPTION	EXPRESSION
`ei_t`	ε_{it}	T	S	ε_{it} velocity normal strain global system	$\frac{\partial u_{it}}{\partial x_i}$
`exy_t`	ε_{xyt}	T	S	ε_{xyt} velocity shear strain global coord. system	$\frac{1}{2}\left(\frac{\partial u_t}{\partial y} + \frac{\partial v_t}{\partial x}\right)$
`en`	ε_n	All	S	nth principal strain, n=1,2,3	
`eni`	ε_{ni}	All	S	nth principal strain component in the x_i direction	
`si`	σ_i	All	S	σ_i normal stress global coord. system	$\sum_{k=1,2,4} D_{ik}(\varepsilon_k)$
`sz`	σ_z	All	S	σ_z normal stress	$\sum_{k=1,2,4} D_{3k}(\varepsilon_k)$
`sxy`	τ_{xy}	All	S	τ_{xy} shear stress global coord. system	$\sum_{k=1,2,4} D_{4k}(\varepsilon_k)$
`si_t`	σ_{it}	T	S	σ_{it} time derivative of normal stress global coord. system	$\sum_{k=1,2,4} D_{ik}\varepsilon_{kt}$
`sz_t`	σ_z	T	S	σ_{zt} time derivative of normal stress	$\sum_{k=1,2,4} D_{3k}\varepsilon_{kt}$
`sxy_t`	τ_{xy}	T	S	τ_{xy} shear stress global coord. system	$\sum_{k=1,2,4} D_{4k}\varepsilon_{kt}$
`sn`	σ_n	All	S	nth principal stress, n=1,2,3	
`sni`	σ_{ni}	All	S	nth principal stress, n=1,2,3 comp. in the x_i direction	

TABLE 8-3: PLANE STRAIN APPLICATION MODE VARIABLES

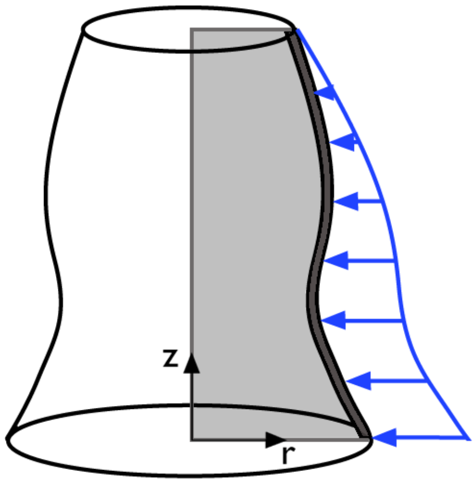
NAME	SYMBOL	ANALYSIS	DOMAIN	DESCRIPTION	EXPRESSION
`tresca`	σ_{tresca}	All	S	Tresca stress	$\max(\max(\lvert\sigma_1-\sigma_2\rvert, \lvert\sigma_2-\sigma_3\rvert)), \lvert\sigma_1-\sigma_3\rvert))$
`mises`	σ_{mises}	All	S	von Mises stress	
`Tai`	Ta_i	All	B	Surface traction (force/area) in x_i direction	$\begin{bmatrix}\mathrm{Ta}_x\\ \mathrm{Ta}_y\end{bmatrix} = \begin{bmatrix}\sigma_x & \tau_{xy}\\ \tau_{xy} & \sigma_y\end{bmatrix}\begin{bmatrix}n_x\\ n_y\end{bmatrix}$
`Fig`	F_{ig}	All	S	Body load in global x_i direction	If force/area $\begin{bmatrix}F_{xg}\\ F_{yg}\end{bmatrix} = \begin{bmatrix}F_x\\ F_y\end{bmatrix}$ If force/volume $\begin{bmatrix}F_{xg}\\ F_{yg}\end{bmatrix} = th\begin{bmatrix}F_x\\ F_y\end{bmatrix}$
`Fig`	F_{ig}	All	B	Edge load in global x_i direction	if force/length $\begin{bmatrix}F_{xg}\\ F_{yg}\end{bmatrix} = \begin{bmatrix}F_x\\ F_y\end{bmatrix}$ if force/area $\begin{bmatrix}F_{xg}\\ F_{yg}\end{bmatrix} = th\begin{bmatrix}F_x\\ F_y\end{bmatrix}$
`Fig`	F_{ig}	All	P	Point load in global x_i direction	$\begin{bmatrix}F_{xg}\\ F_{yg}\end{bmatrix} = \begin{bmatrix}F_x\\ F_y\end{bmatrix}$

Note: To form the complete application mode variable names, add a suffix consisting of an underscore and the application mode name (default: pn), for example, `mises_pn`. (This does not apply to the dependent variables for the displacements.)

The Axial Symmetry, Stress-Strain Application Mode

The Axial Symmetry, Stress-Strain application mode uses the cylindrical coordinates r, φ, and z. It solves the equations for the global displacement (u, w) in the r and z directions. The displacement v in the φ direction together with the $\tau_{r\varphi}$, $\tau_{\varphi z}$, $\gamma_{r\varphi}$, and $\gamma_{\varphi z}$ components of the stresses and strains are assumed to be zero. In this mode, loads are independent of φ, and it allows them only in the r and z directions.

You can view the domain where the equations are solved as the intersection between the original axisymmetric 3D solid and the half plane $\varphi = 0$, $r \geq 0$. Therefore it is necessary to draw the geometry only in the half plane $r \geq 0$. Later on, recover the original 3D solid by rotating the 2D geometry about the z-axis (see the figure below).



Loads in an axisymmetric stress-strain analysis. The modeling domain is the gray 2D section.

The equilibrium equations in axial symmetry read

$$\frac{\partial\sigma_r}{\partial r} + \frac{\partial\tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + K_r = 0 \qquad (8\text{-}2)$$
$$\frac{\partial\tau_{rz}}{\partial r} + \frac{\partial\sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + K_z = 0$$

The strain-displacement relationships for the axial symmetry case for small displacements are

$$\varepsilon_r = \frac{\partial u}{\partial r} \qquad \varepsilon_\varphi = \frac{u}{r} \qquad \varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

To avoid division by r in the equilibrium equations (which causes problems on the axis, where $r = 0$), the COMSOL Multiphysics application mode transforms the equations. It multiplies the first equation by r^2 and the second by r. This transformation is a natural choice because it appears in the principle of virtual work. Integrating over the volume, you must multiply the integrand by $2\pi r$. The application mode introduces and solves for a new dependent variable

$$\mathrm{uor} = \frac{u}{r}$$

which replaces the true radial displacement u.

Note: $r = 0$ is the symmetry axis. In the Axial Symmetry, Stress-Strain application mode the r-axis is in the x direction (horizontal), and the z-axis is in the y direction (vertical).

PDE Formulation

COMSOL Multiphysics formulates this application mode by using the equilibrium equation (see Equation 8-2) described in general terms in the section "Implementation" on page 199.

APPLICATION MODE PARAMETERS

The application mode parameters defining the loads, material, and constraints are explained in the section "Application Mode Descriptions" on page 202.

Application Mode Variables

A large number of variables are available for use in expressions and for postprocessing. In addition to the variables listed in Table 8-4, almost all application mode parameters are available as variables. Some vary for different analyses as you can see in the Analysis column.

ANALYSIS	ABBREVIATION
Static	S
Time dependent	T

TABLE 8-4: AXIAL SYMMETRY STRESS-STRAIN APPLICATION MODE VARIABLES

NAME	SYMBOL	ANALYSIS	DOMAIN	DESCRIPTION	EXPRESSION
`uor`	uor	All	All	r displacement divided by r	uor
`uaxi`	uaxi	All	All	r displacement	$\mathrm{uor}\cdot r$
`w`	w	All	All	z displacement	w
`uor_t`	uor_t	T	All	r velocity divided by r	uor_t
`uaxi_t`	uaxi_t	T	All	r velocity	$\mathrm{uor}_t \cdot r$
`w_t`	w_t	T	All	z velocity	w_t
`disp`	disp	All	All	Total displacement	$\sqrt{\mathrm{uaxi}^2 + w^2}$
`er`	ε_r	All	S	ε_r normal strain global system	$\mathrm{uor} + \frac{\partial}{\partial r}(\mathrm{uor}) \cdot r$
`ez`	ε_z	All	S	ε_z normal strain global system	$\frac{\partial w}{\partial z}$
`ephi`	ε_φ	All	S	ε_φ normal strain	uor
`erz`	ε_{rz}	All	S	ε_{rz} shear strain global coord. system	$\frac{1}{2}\left(\frac{\partial}{\partial z}(\mathrm{uor}) \cdot r + \frac{\partial w}{\partial r}\right)$
`er_t`	ε_{rt}	T	S	ε_{rt} velocity normal strain global system	$\mathrm{uor}_t + \frac{\partial}{\partial r}(\mathrm{uor}_t) \cdot r$
`ez_t`	ε_{zt}	All	S	ε_{zt} velocity normal strain global system	$\frac{\partial w_t}{\partial z}$
`ephi_t`	$\varepsilon_{\varphi t}$	All	S	$\varepsilon_{\varphi t}$ velocity normal strain	uor_t

TABLE 8-4: AXIAL SYMMETRY STRESS-STRAIN APPLICATION MODE VARIABLES

NAME	SYMBOL	ANALYSIS	DOMAIN	DESCRIPTION	EXPRESSION
`erz_t`	ε_{rzt}	All	S	ε_{rzt} shear strain global coord. system	$\frac{1}{2}\left(\frac{\partial}{\partial z}(\mathrm{uor}_t)\cdot r+\frac{\partial w_t}{\partial r}\right)$
`en`	ε_n	All	S	n st/nd principal strain, n = 1, 2, 3	
`eni`	ε_{ni}	All	S	n st/nd principal strain component in the x_i direction	
`sr`	σ_r	All	S	σ_r normal stress global coord. system	$\sum_k D_{1k}(\varepsilon_k)$
`sphi`	σ_φ	All	S	σ_φ normal stress	$\sum_k D_{2k}(\varepsilon_k)$
`sz`	σ_z	All	S	σ_z normal stress global coord. system	$\sum_k D_{3k}(\varepsilon_k)$
`srz`	τ_{rz}	All	S	τ_{rz} shear stress global coord. system	$\sum_{k=1,2,4} D_{4k}(\varepsilon_k)$
`sr_t`	σ_{rt}	All	S	σ_{rt} time der. of normal stress global coord. system	$\sum_k D_{1k}\varepsilon_{kt}$
`sphi_t`	$\sigma_{\varphi t}$	All	S	$\sigma_{\varphi t}$ time der. of normal stress	$\sum_k D_{2k}\varepsilon_{kt}$
`sz_t`	σ_{zt}	All	S	σ_{zt} time der. of normal stress global coord. system	$\sum_k D_{3k}\varepsilon_{kt}$
`srz_t`	τ_{rzt}	All	S	τ_{rzt} time der. of shear stress global coord. system	$\sum_{k=1,2,4} D_{4k}\varepsilon_{kt}$
`sn`	σ_n	All	S	n st/nd principal stress, n=1,2,3	

TABLE 8-4: AXIAL SYMMETRY STRESS-STRAIN APPLICATION MODE VARIABLES

NAME	SYMBOL	ANALYSIS	DOMAIN	DESCRIPTION	EXPRESSION
`sni`	σ_{ni}	All	S	nth principal stress, n=1,2,3 component in the x_i direction	
`tresca`	σ_{tresca}	All	S	Tresca stress	$\mathrm{max}(\mathrm{max}(\lvert\sigma_1-\sigma_2\rvert, \lvert\sigma_2-\sigma_3\rvert)), \lvert\sigma_1-\sigma_3\rvert))$
`mises`	σ_{mises}	All	S	von Mises stress	
`Tar, Taz`	$\mathrm{Ta}_r, \mathrm{Ta}_z$	All	B	Surface traction (force/area) in r and z direction	$\begin{bmatrix}\mathrm{Ta}_r\\ \mathrm{Ta}_z\end{bmatrix} = \begin{bmatrix}\sigma_r & \tau_{rz}\\ \tau_{rz} & \sigma_z\end{bmatrix}\begin{bmatrix}n_r\\ n_z\end{bmatrix}$
`Frg, Fzg`	F_{rg}, F_{zg}	All	All	Body, edge, point load in global r and z directions	$\begin{bmatrix}F_{rg}\\ F_{zg}\end{bmatrix} = \begin{bmatrix}F_r\\ F_z\end{bmatrix}$

Note: To form the complete application mode variable names, add a suffix consisting of an underscore and the application mode name (default: `axi`), for example, `mises_axi`. (This does not apply to the dependent variables for the displacements.)

Examples of Structural Mechanics Models

Two structural mechanics benchmark models show how to perform a linear static stress analysis using:

- A uniformly distributed horizontal load along an outer edge
- A gravity load

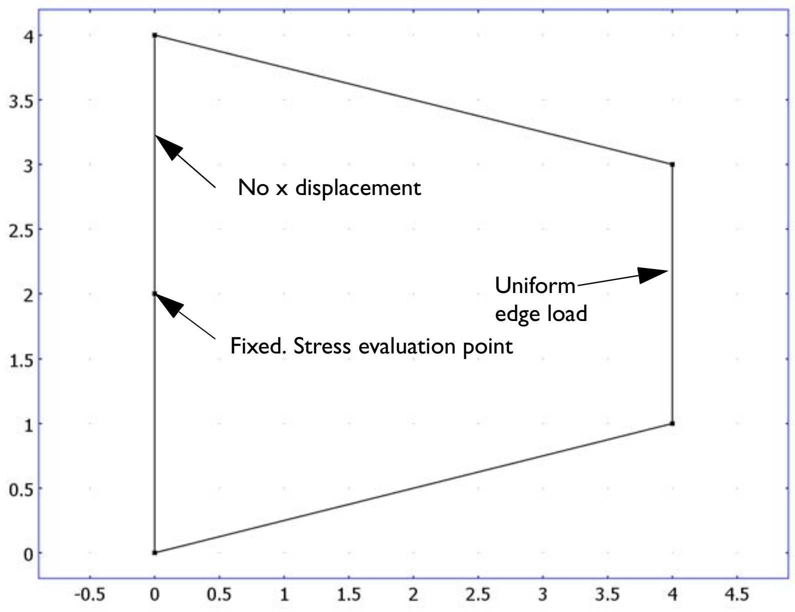
The examples are taken from a NAFEMS benchmark collection (Ref. 1).

Tapered Membrane End Load

The first example shows a 2D plane stress model of a membrane with a thickness of 0.1 m. The load is a uniformly distributed horizontal load of 10 MN/m (that is, a pressure of 100 MPa) along the right end. At the left end, there is no displacement in the *x* direction. Also, at a midpoint location, the left end is fixed in the *y* direction.

The model uses the following material properties:

- The material is isotropic.
- The Young's modulus (elasticity modulus) is $210 \cdot 10^3$ MPa.
- The Poisson ratio is 0.3.



Modeling in COMSOL Multiphysics

Using a Plane Stress application modes and a static analysis, it is straightforward to perform this stress analysis. The finite element model uses second-order triangular Lagrange elements. To show convergence toward the benchmark value, refine the mesh and recompute the solution twice.

Results

The solution shows an x-direction stress at the point $(0, 2)$ that is in good agreement with the benchmark target value of 61.3 MPa. Using the initial mesh, the COMSOL Multiphysics solution gives a value of 61.41 MPa. Two successive mesh refinements provide stress values of 61.36 MPa and 61.35 MPa.

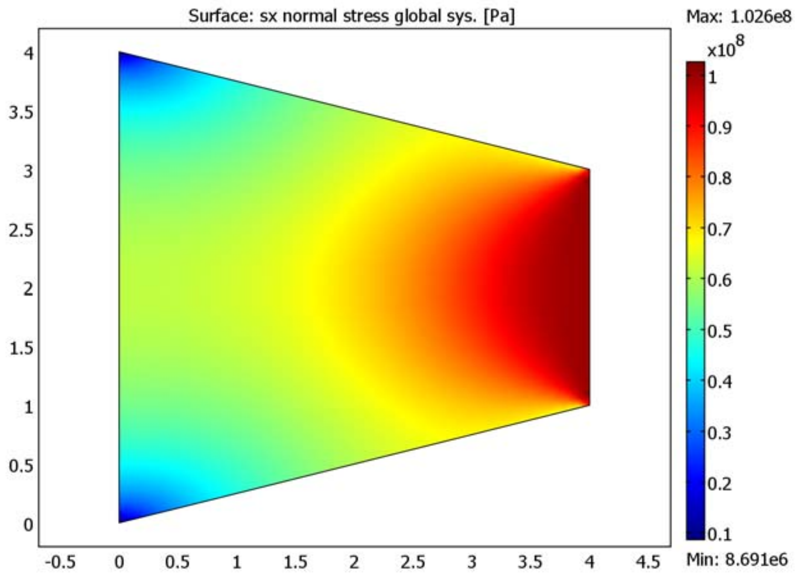


Figure 8-1: The x-direction stress distribution with a uniform edge load

Model Library path: `COMSOL_Multiphysics/Structural_Mechanics/edge_load_2d`

Modeling Using the Graphical User Interface

MODEL NAVIGATOR

1 Select **2D** in the **Space dimension** list.
2 In the list of application modes, open the **COMSOL Multiphysics>Structural Mechanics** folder and then the **Plane Stress** node. Select **Static analysis**.
3 Click **OK**.

GEOMETRY MODELING

1 On the **Draw** menu, point to **Specify Objects** and then click **Line**.

2 In the **Line** dialog box, type `0 4 4 0 0` in the **x** edit field and `0 1 3 4 0` in the **y** edit field.

3 Click **OK**.

4 Click the **Zoom Extents** button on the Main toolbar.

5 Click the **Coerce to Solid** button on the Draw toolbar.

6 On the **Draw** menu, point to **Specify Objects** and then click **Point**.

7 In the **Point** dialog box, type `0` in the **x** edit field and `2` in the **y** edit field.

8 Click **OK**.

The point is the location for the point constraint and also the benchmark value for the stress.

PHYSICS SETTINGS

Boundary and Point Conditions—Loads and Constraints

1 From the **Physics** menu, choose **Boundary Settings**.

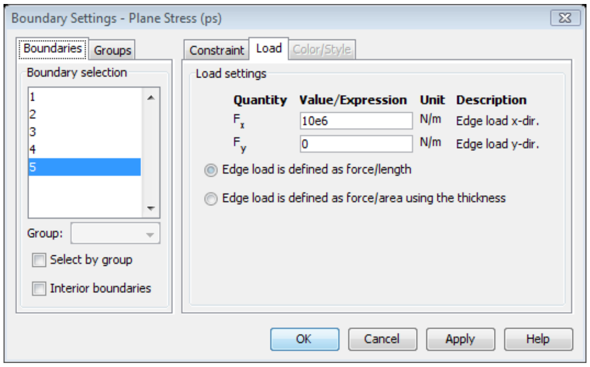
2 Select Boundaries 1 and 3 from the **Boundary selection** list.

3 Select the $\mathbf{R_x}$ check box.

4 Click the **Load** tab.

5 Select Boundary 5 in the **Boundary selection** list.

6 Type `10e6` in the $\mathbf{F_x}$ edit field.


Boundary Settings - Plane Stress (ps)

Boundaries | Groups

Boundary selection: 1, 2, 3, 4, 5 (5 selected)

Group:

Select by group

Interior boundaries

Constraint | Load | Color/Style

Load settings

Quantity	Value/Expression	Unit	Description
F_x	10e6	N/m	Edge load x-dir.
F_y	0	N/m	Edge load y-dir.

Edge load is defined as force/length

Edge load is defined as force/area using the thickness

OK | Cancel | Apply | Help

7 Click **OK**.

8 From the **Physics** menu, choose **Point Settings**.

9 Select Point 2 from the **Point selection** list.

10 Select the $\mathbf{R_x}$ and $\mathbf{R_y}$ check boxes.

11 Click **OK**.

Subdomain Settings—Material Properties

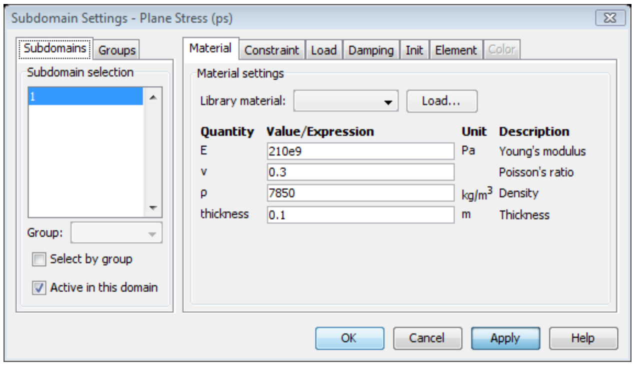
1 From the **Physics** menu, choose **Subdomain Settings**.

2 Select Subdomain 1 from the **Subdomain selection** list.

3 Type `210e9` in the **E** edit field for the Young's modulus.

4 Type `0.3` in the ν edit field for the Poisson's modulus.

5 Type `0.1` in the **thickness** edit field.



Quantity	Value/Expression	Unit	Description
E	210e9	Pa	Young's modulus
ν	0.3		Poisson's ratio
ρ	7850	kg/m^3	Density
thickness	0.1	m	Thickness

6 Click **OK**.

The other material properties are not used in this model.

MESH GENERATION

Initialize an unstructured triangular mesh and refine it once:

1 Click the **Initialize Mesh** button on the Main toolbar.

2 Click the **Refine Mesh** button on the Main toolbar.

COMPUTING THE SOLUTION

Click the **Solve** button on the Main toolbar.

POSTPROCESSING AND VISUALIZATION

The default plot shows the von Mises stress. To plot the x-direction stress, use the following steps:

1 From the **Postprocessing** menu, select **Plot Parameters**.

2 In the **Plot Parameters** dialog box, click the **Surface** tab.

3 On the **Surface Data** tab, select **sx normal stress global sys.** from the **Predefined quantities** list.

4 Click **OK**.

To get a better look at the value of the x-direction stress at the point (0,2), use the **Cross-Section Plot Parameters** dialog box:

1 From the **Postprocessing** menu, select **Cross-Section Plot Parameters**.

2 Click the **Point** tab.

3 Select **sx normal stress global sys.** from the **Predefined quantities** list.

4 Type 0 in the **x** edit field and 2 in the **y** edit field.

5 Click **OK**.

This plot shows a straight line representing the value of the x-direction stress at the point (0, 2). You can also click at that point in the results plot. That prints the numeric value in the message log.

Tapered Cantilever Gravity Load

The second example shows another 2D plane stress model of a membrane with a thickness of 0.1 m. The load is a gravity load that acts in the negative y direction with an acceleration of 9.81 m/s^2. The left end boundary is fully fixed (no displacements).

The model uses the following material properties:

- The material is isotropic.
- The Young's modulus (elasticity modulus) is $210 \cdot 10^3$ MPa.
- The Poisson ratio is 0.3.
- The density is 7000 kg/m^3.

Modeling in COMSOL Multiphysics

Using a Plane Stress application modes and a static analysis, it is straightforward to perform this stress analysis. You enter the gravity load as force/volume. COMSOL Multiphysics then computes the load using the thickness of the material. The finite element model uses second-order triangular Lagrange elements. For postprocessing, use a cross-section plot to show the value of the shear stress at the location (0,2).

Results

The solution shows a shear stress (s_{xy}) at the point (0,2) that is in good agreement with the benchmark target value of −0.200 MPa. Using the initial mesh, the COMSOL Multiphysics solution gives a value of −0.199 MPa.

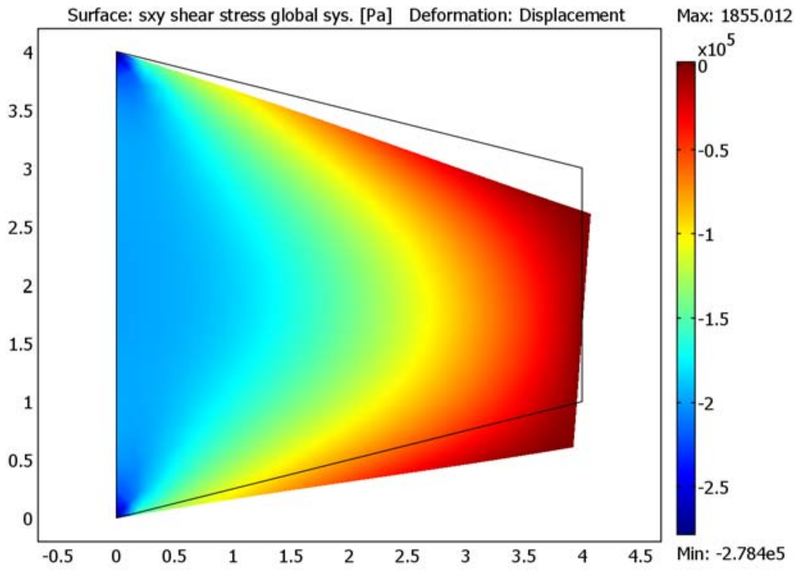


Figure 8-2: The shear stress and the displacement (exaggerated) from a gravity load.

Model Library path: `COMSOL_Multiphysics/Structural_Mechanics/gravity_load_2d`

Modeling Using the Graphical User Interface

MODEL NAVIGATOR

1 Select **2D** from the **Space dimension** list.

2 In the list of application modes, open the **COMSOL Multiphysics>Structural Mechanics** folder and then the **Plane Stress** node. Select **Static analysis**.

3 Click **OK**.

GEOMETRY MODELING

1 On the **Draw** menu, point to **Specify Objects** and then click **Line**.

2 In the **Line** dialog box, type `0 4 4 0 0` in the **x** edit field and `0 1 3 4 0` in the **y** edit field.

3 Click **OK**.

4 Click the **Zoom Extents** button on the Main toolbar.

5 Click the **Coerce to Solid** button on the Draw toolbar.

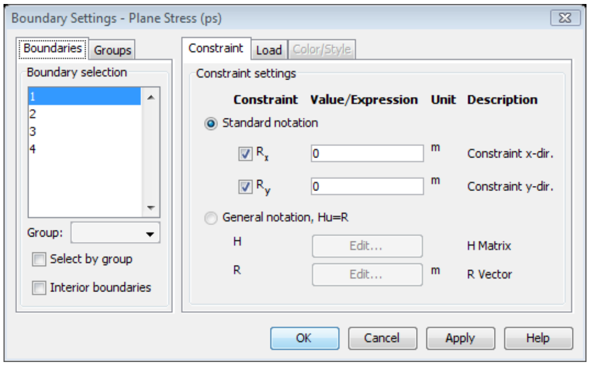
PHYSICS SETTINGS

Boundary Conditions—Constraints

1 From the **Physics** menu, choose **Boundary Settings**.

2 Select Boundary 1 in the **Boundary selection** list.

3 Select the **R_x** and **R_y** check boxes.


Boundary Settings - Plane Stress (ps)

Boundary selection: 1 (selected), 2, 3, 4

Group: Select by group [ ], Interior boundaries [ ]

Constraint settings

Constraint	Value/Expression	Unit	Description
● Standard notation			
✓ R_x	0	m	Constraint x-dir.
✓ R_y	0	m	Constraint y-dir.
○ General notation, Hu=R			
H	Edit...		H Matrix
R	Edit...	m	R Vector

4 Click **OK**.

Subdomain Settings—Material Properties and Loads

1 From the **Physics** menu, choose **Subdomain Settings**.

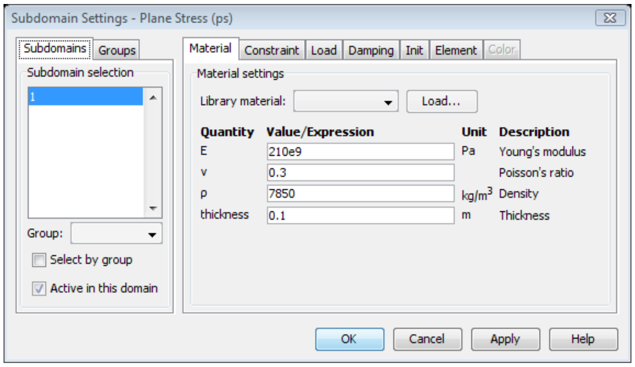
2 Select Subdomain 1 in the **Subdomain selection** list.

3 Type `210e9` in the **E** edit field for the Young's modulus.

4 Type `0.3` in the ν edit field for the Poisson's ratio.

5 Type `0.1` in the **thickness** edit field.

You can also type `7000` in the ρ edit field for the density, but the density value from this edit field is not used in the static analysis. The gravity appears in the specification of the gravity load below.



Subdomain Settings - Plane Stress (ps)

Material settings

Library material:

Quantity	Value/Expression	Unit	Description
E	210e9	Pa	Young's modulus
ν	0.3		Poisson's ratio
ρ	7850	kg/m^3	Density
thickness	0.1	m	Thickness

Subdomain selection: 1

Group:

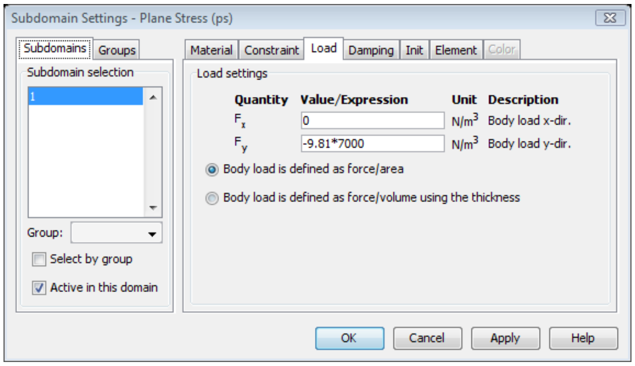
Select by group

[✓ Active in this domain]

6 Click the **Load** tab.

7 Type `-9.81*7000` in the $\mathbf{F_y}$ edit field for the *y*-direction body load.

8 Click the **Body load is defined as force/volume using the thickness** button.



Subdomain Settings - Plane Stress (ps)

Load settings

Quantity	Value/Expression	Unit	Description
F_x	0	N/m^3	Body load x-dir.
F_y	-9.81*7000	N/m^3	Body load y-dir.

[● Body load is defined as force/area]

Body load is defined as force/volume using the thickness

Subdomain selection: 1

Group:

Select by group

[✓ Active in this domain]

9 Click **OK**.

MESH GENERATION

Click the **Initialize Mesh** button on the Main toolbar.

COMPUTING THE SOLUTION

Click the **Solve** button on the Main toolbar.

POSTPROCESSING AND VISUALIZATION

The default plot shows the von Mises stress. To plot the shear stress and also the deformed shape (displacements), use the following steps:

1. From the **Postprocessing** menu, select **Plot Parameters**.
2. In the **Plot Parameters** dialog box, click the **Surface** tab.
3. On the **Surface Data** tab, select **sxy shear stress global sys.** from the **Predefined quantities** list.
4. Click the **Deform** tab.
5. Select the **Deformed shape plot** check box. The deformation data is the *x* and *y* displacements by default.
6. Click **OK**.

The deformed shape plot uses a scaling factor to clearly show the deformation. The actual displacements are small compared to the model geometry. Click the **Deform** tab too see the scale factor or to enter a different scale factor.

To get a better look at the value of the shear stress at the point (0, 2), use the **Cross-Section Plot Parameters** dialog box:

1. From the **Postprocessing** menu, select **Cross-Section Plot Parameters**.
2. Click the **Point** tab.
3. Select **sxy shear stress global sys.** from the **Predefined quantities** list.
4. Type 0 in the **x** edit field and 2 in the **y** edit field.
5. Click **OK**.

The plot shows a straight line at the value of the shear stress at (0, 2). You can also click at that point in the results plot. That prints the numeric value in the message log.

Reference

1. D. Hitchings, A. Kamoulakos, G. A. O. Davies: *Linear Statics Benchmarks Vol. 1*, NAFEMS Ltd., Glasgow, 1987.

9

PDE Modes for Equation-Based Modeling

This chapter describes the use of the PDE modes for equation-based modeling. A step-by-step example illustrates the use of the PDE modes for solving Poisson's equation.

The PDE Modes

The PDE modes are application modes for equation-based modeling. They support three types of PDE formulation:

- *Coefficient form* for linear or almost linear PDEs, explained in detail in the section "Using the Coefficient Form PDEs" on page 239.
- *General form* for nonlinear PDEs. You find a detailed discussion about the general form in the section "Using the General Form PDE" on page 249.
- *Weak form* using the weak formulation of the PDE for maximum flexibility. For more information about the weak form, see "The Weak Form" on page 291.

In addition, a number of specialized instances of these PDE formulations provide interfaces for a number of classical PDEs. See "Classical PDEs" on page 260 for more information.

The following table lists the available PDE modes with their default application mode names:

TABLE 9-1: DEFAULT APPLICATION MODE NAMES FOR THE PDE MODES

PDE MODE	APPLICATION MODE NAME
PDE, Coefficient Form	c
PDE, General Form	g
Weak Form, Subdomain	w
Weak Form, Boundary	wb
Weak Form, Edge	we
Weak Form, Point	wp
Convection-Diffusion Equation	cdeq
Heat Equation	hteq
Helmholtz Equation	hzeq
Laplace's Equation	lpeq
Poisson's Equation	poeq
Schrödinger Equation	scheq
Wave Equation	waeq

Using the PDE Modes

Starting a Model Using a PDE Mode

Do as follows to create a new model using one of the PDE modes:

1 Start COMSOL Multiphysics or click **New** in the COMSOL Multiphysics window.

2 In the **Model Navigator** make a selection from the **Space dimension** list.

3 Go to the folder **Application Modes>COMSOL Multiphysics>PDE Modes**.

4 Open either the **PDE, Coefficient Form** folder, the **PDE, General Form** folder, or one of the weak form folders, and make a further selection between the analysis types available.

   Alternatively, open the **Classical PDEs** folder to select from a list of classical PDEs, which are all special cases of the coefficient form formulation.

5 Click **OK**.

SPECIFYING A SYSTEM OF PDES

COMSOL Multiphysics allows the creation of PDEs with more than one dependent variable. To add variables, follow steps 1 through 4 above and then enter all the variable names as space-separated entries in the **Dependent variables** edit field before clicking **OK**. (For the classical PDEs, you can change the name of the dependent variable but not add new ones.) The number of equations in the model equals the number of dependent variables. For example, to start a model with three dependent variables u_1, u_2, and u_3, enter:

```
u1 u2 u3
```

You can also couple several scalar PDEs using a multiphysics approach.

Using the Coefficient Form PDEs

The coefficient form PDE mode covers many well-known PDEs. This section covers the formulation and settings pertaining to the coefficient form, as well as the general PDE terminology used in COMSOL Multiphysics.

THE SCALAR COEFFICIENT FORM EQUATION

A single dependent variable u is an unknown function on the computational domain. COMSOL Multiphysics determines it by solving the PDE problem that you specify. In coefficient form, the PDE problem reads

$$\begin{cases} e_a \dfrac{\partial^2 u}{\partial t^2} + d_a \dfrac{\partial u}{\partial t} + \nabla \cdot (-c\nabla u - \alpha u + \gamma) + \beta \cdot \nabla u + au = f & \text{in } \Omega \\ \mathbf{n} \cdot (c\nabla u + \alpha u - \gamma) + qu = g - h^T \mu & \text{on } \partial\Omega \\ hu = r & \text{on } \partial\Omega \end{cases} \tag{9-1}$$

where

- Ω is the computational domain—the union of all subdomains
- $\partial\Omega$ is the domain boundary
- $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega$

The first equation in the list above is the PDE, which must be satisfied in Ω. The second and third equations are the boundary conditions, which must hold on $\partial\Omega$. The second equation is a *generalized Neumann* boundary condition, whereas the third equation is a *Dirichlet* boundary condition. This nomenclature and the second equation above deviate slightly from traditional usage in potential theory where a Neumann condition usually refers to the case $q = 0$. The generalized Neumann condition is also called a *mixed boundary condition* or a *Robin boundary condition*. In finite element terminology, Neumann boundary conditions are called *natural boundary conditions* because they do not occur explicitly in the weak form of the PDE problem. Dirichlet conditions are called *essential boundary conditions* because they restrict the trial space. Dirichlet boundary conditions often represent *constraints*.

This manual uses the following conventions:

- The symbol ∇ is the vector differential operator (gradient), defined as

  $$\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$$

  The space coordinates are denoted $x_1, \dots, x_n$, where n represents the number of space dimensions.
- The symbol Δ stands for the Laplace operator

$$\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$$

- $\nabla \cdot (c \nabla u)$ means

$$\frac{\partial}{\partial x_1}\left(c\frac{\partial u}{\partial x_1}\right) + \ldots + \frac{\partial}{\partial x_n}\left(c\frac{\partial u}{\partial x_n}\right)$$

- $\beta \cdot \nabla u$ means

$$\beta_1 \frac{\partial u}{\partial x_1} + \ldots + \beta_n \frac{\partial u}{\partial x_n}$$

where $\beta_1, \ldots, \beta_n$ are the components of the vector β.

Within COMSOL Multiphysics, you specify the coefficients c, α, γ, β, a, q, and h, and the terms f, g, and r. They can all be functions of the spatial coordinates.

- A PDE is *linear* when the coefficients depend only on the spatial coordinates (or are constants).
- A PDE is *nonlinear* if the coefficients depend on u or its derivatives (the components of ∇u).
- All the coefficients in the above equation are scalars except α, β, and γ, which are vectors with n components. The coefficient c can alternatively be an n-by-n matrix to model anisotropic materials. For more information see "Modeling Anisotropic Material" on page 215 in the *COMSOL Multiphysics User's Guide*

The e_a coefficient in Equation 9-1 is a scalar or a matrix for time-dependent systems called the *mass matrix* (or mass coefficient). The d_a coefficient represents a damping term (however, if $e_a = 0$, then d_a is often called the mass coefficient). See "Solving Time-Dependent Problems" on page 256 for more information on time-dependent problems.

Interpreting Boundary Conditions

The formulation of the boundary conditions imposes both Dirichlet and Neumann conditions. This combination is possible because of a new dependent variable μ, which is defined only on the boundary. The unknown variable μ is called a *Lagrange multiplier*. Often you can reformulate boundary conditions without Lagrange multipliers. In structural mechanics problems the Lagrange multiplier equals the reaction forces on the boundary. The factor h^T in the Neumann boundary condition is the transpose of h. If h is a scalar, then $h^T = h$.

How do the Lagrange multipliers relate to the conventional formulation with either a Dirichlet or a Neumann boundary condition?

- First, assume that $h = 1$. Then the Dirichlet condition is $u = r$. The Neumann condition becomes:

$$\mathbf{n} \cdot (c\nabla u + \alpha u - \gamma) + qu = g - \mu$$

  The Lagrange multiplier, μ, adjusts so as to satisfy the requested Dirichlet condition. Specifying a nonzero g or q changes the value of the Lagrange multiplier but does not affect the actual solution u. Therefore, this equation can usually be ignored, leaving a pure Dirichlet condition.

- Second, assume that $h = 0$ and $r = 0$. Then the Dirichlet condition reads $0 = 0$, and the Neumann condition is

$$\mathbf{n} \cdot (c\nabla u + \alpha u - \gamma) + qu = g$$

  This is the generalized Neumann condition without a Lagrange multiplier. For more on the Lagrange multiplier formulation, see "System for Two Variables in the General Form" on page 251.

The vector $\Gamma = -c\nabla u - \alpha u + \gamma$ is the *flux vector*. In transport equations its first term describes diffusion, the second term describes convection with a velocity $-\alpha$, and the third term γ is a source term.

In certain applications Γ can be discontinuous across interior boundaries. There can be a jump in the normal component of Γ across such a boundary. For instance, denote the two adjacent subdomains as 1 and 2, and let Γ_i and $\mathbf{n}_i$ be the values of Γ and $\mathbf{n}$ from the two subdomains. Then you can state the jump condition as the Neumann condition

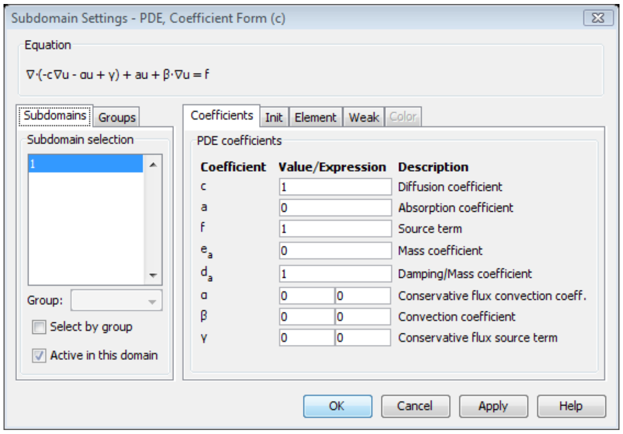
$$-\mathbf{n}_1 \cdot \Gamma_1 - \mathbf{n}_2 \cdot \Gamma_2 + qu = g - h^T\mu$$

where $\mathbf{n}_1$ is the outward normal from Subdomain 1, and $\mathbf{n}_2$ is the outward normal from Subdomain 2 so that $\mathbf{n}_1 = -\mathbf{n}_2$. At the same time there is a Dirichlet condition $hu = r$. (Let $h = r = 0$ to discard the Dirichlet condition and the Lagrange multiplier μ.) To specify such conditions on interior boundaries, select the **Enable Interior Boundaries** check box in the **Boundary Settings** dialog box.

The rest of this section handles stationary problems where $\partial u / \partial t = 0$ and transient effects have vanished.

Specifying PDE Coefficients on Subdomains

To specify PDE coefficients, open the **Subdomain Settings** dialog box for a stationary, scalar 2D PDE problem in coefficient form. From the **Physics** menu, choose **Subdomain Settings**.



Subdomain Settings - PDE, Coefficient Form (c)

Equation

$\nabla\cdot(-c\nabla u - \alpha u + \gamma) + au + \beta\cdot\nabla u = f$

Subdomains | Groups

Subdomain selection: 1

Group:

Select by group

[✓] Active in this domain

Coefficients | Init | Element | Weak | Color

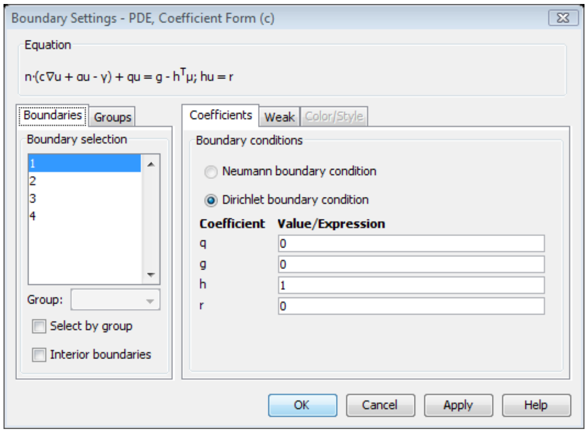
PDE coefficients

Coefficient	Value/Expression		Description
c	1		Diffusion coefficient
a	0		Absorption coefficient
f	1		Source term
e_a	0		Mass coefficient
d_a	1		Damping/Mass coefficient
α	0	0	Conservative flux convection coeff.
β	0	0	Convection coefficient
γ	0	0	Conservative flux source term

The default values correspond to Poisson’s equation with a source value of 1.

Specifying Boundary Conditions

To specify boundary conditions, open the **Boundary Settings** dialog box. From the **Physics** menu, choose **Boundary Settings**.



Boundary Settings - PDE, Coefficient Form (c)

Equation

$n\cdot(c\nabla u + \alpha u - \gamma) + qu = g - h^T\mu;\ hu = r$

Boundaries | Groups

Boundary selection: 1, 2, 3, 4

Group:

Select by group

Interior boundaries

Coefficients | Weak | Color/Style

Boundary conditions

Neumann boundary condition

[✓ Dirichlet boundary condition]

Coefficient	Value/Expression
q	0
g	0
h	1
r	0

The default values correspond to homogeneous Dirichlet boundary conditions.

To specify boundary conditions:

1 Make a selection from the **Boundary selection** list.

2 Select a Neumann or Dirichlet boundary condition type.

3 Type the values for the coefficients that are active for the boundary condition type.

Note: The c, α, and γ coefficients in the equations for the boundary conditions come from the PDE specification in the **Subdomain Settings** dialog box.

THE COEFFICIENT FORM EQUATION SYSTEM

With two independent variables u_1 and u_2, the stationary PDE problem in coefficient form results in the following equation system:

$$e_a \frac{\partial^2 u}{\partial t^2} + d_a \frac{\partial u}{\partial t} - \nabla \cdot (c\nabla u + \alpha u - \gamma) + \beta \cdot \nabla u + au = f$$

where $u = (u_1, u_2)$. The mass term is defined as

$$e_a \frac{\partial^2 u}{\partial t^2} = \begin{bmatrix} e_{a11} & e_{a12} \\ e_{a21} & e_{a22} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 u_1}{\partial t^2} \\ \frac{\partial^2 u_2}{\partial t^2} \end{bmatrix} = \begin{bmatrix} e_{a11}\frac{\partial^2 u_1}{\partial t^2} + e_{a12}\frac{\partial^2 u_2}{\partial t^2} \\ e_{a21}\frac{\partial^2 u_1}{\partial t^2} + e_{a22}\frac{\partial^2 u_2}{\partial t^2} \end{bmatrix}$$

Similarly, the damping term is

$$d_a \frac{\partial u}{\partial t} = \begin{bmatrix} d_{a11} & d_{a12} \\ d_{a21} & d_{a22} \end{bmatrix} \begin{bmatrix} \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial t} \end{bmatrix} = \begin{bmatrix} d_{a11}\frac{\partial u_1}{\partial t} + d_{a12}\frac{\partial u_2}{\partial t} \\ d_{a21}\frac{\partial u_1}{\partial t} + d_{a22}\frac{\partial u_2}{\partial t} \end{bmatrix}$$

However, if $e_a = 0$, then d_a is often called the mass coefficient.

The diffusive flux is defined as

$$c\nabla u = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \nabla \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \nabla u_1 \\ \nabla u_2 \end{bmatrix} = \begin{bmatrix} c_{11}\nabla u_1 + c_{12}\nabla u_2 \\ c_{21}\nabla u_1 + c_{22}\nabla u_2 \end{bmatrix} = \begin{bmatrix} c_{11}\begin{bmatrix} \frac{\partial u_1}{\partial x} \\ \frac{\partial u_1}{\partial y} \end{bmatrix} + c_{12}\begin{bmatrix} \frac{\partial u_2}{\partial x} \\ \frac{\partial u_2}{\partial y} \end{bmatrix} \\ c_{21}\begin{bmatrix} \frac{\partial u_1}{\partial x} \\ \frac{\partial u_1}{\partial y} \end{bmatrix} + c_{22}\begin{bmatrix} \frac{\partial u_2}{\partial x} \\ \frac{\partial u_2}{\partial y} \end{bmatrix} \end{bmatrix} =$$

$$= \begin{bmatrix} \begin{bmatrix} c_{11}\frac{\partial u_1}{\partial x} + c_{12}\frac{\partial u_2}{\partial x} \\ c_{11}\frac{\partial u_1}{\partial y} + c_{12}\frac{\partial u_2}{\partial y} \end{bmatrix} \\ \begin{bmatrix} c_{21}\frac{\partial u_1}{\partial x} + c_{22}\frac{\partial u_2}{\partial x} \\ c_{21}\frac{\partial u_1}{\partial y} + c_{22}\frac{\partial u_2}{\partial y} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \mathrm{cu1x} \\ \mathrm{cu1y} \end{bmatrix} \\ \begin{bmatrix} \mathrm{cu2x} \\ \mathrm{cu2y} \end{bmatrix} \end{bmatrix}$$

where ∇u_1 and ∇u_2 are column vectors. The flux matrix or flux tensor is a column vector in this presentation. For anisotropic materials, each of the components c_{11}, c_{12}, c_{21}, and c_{22} can be matrices as described above for the one-variable coefficient form PDE. In this case, the diffusive flux reads

$$c\nabla u = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \nabla u_1 \\ \nabla u_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} c_{1111} & c_{1112} \\ c_{1121} & c_{1122} \end{bmatrix} & \begin{bmatrix} c_{1211} & c_{1212} \\ c_{1221} & c_{1222} \end{bmatrix} \\ \begin{bmatrix} c_{2111} & c_{2112} \\ c_{2121} & c_{2122} \end{bmatrix} & \begin{bmatrix} c_{2211} & c_{2212} \\ c_{2221} & c_{2222} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \nabla u_1 \\ \nabla u_2 \end{bmatrix} =$$

$$\begin{bmatrix} \begin{bmatrix} c_{1111} & c_{1112} \\ c_{1121} & c_{1122} \end{bmatrix}\nabla u_1 + \begin{bmatrix} c_{1211} & c_{1212} \\ c_{1221} & c_{1222} \end{bmatrix}\nabla u_2 \\ \begin{bmatrix} c_{2111} & c_{2112} \\ c_{2121} & c_{2122} \end{bmatrix}\nabla u_1 + \begin{bmatrix} c_{2211} & c_{2212} \\ c_{2221} & c_{2222} \end{bmatrix}\nabla u_2 \end{bmatrix}$$

The conservative convective flux is defined as

$$\alpha u = \alpha\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \alpha_{111} \\ \alpha_{112} \end{bmatrix} & \begin{bmatrix} \alpha_{121} \\ \alpha_{122} \end{bmatrix} \\ \begin{bmatrix} \alpha_{211} \\ \alpha_{212} \end{bmatrix} & \begin{bmatrix} \alpha_{221} \\ \alpha_{222} \end{bmatrix} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \alpha_{111} \\ \alpha_{112} \end{bmatrix} u_1 + \begin{bmatrix} \alpha_{121} \\ \alpha_{122} \end{bmatrix} u_2 \\ \begin{bmatrix} \alpha_{211} \\ \alpha_{212} \end{bmatrix} u_1 + \begin{bmatrix} \alpha_{221} \\ \alpha_{222} \end{bmatrix} u_2 \end{bmatrix}$$

Here the third index, k, of α_{ijk} corresponds to the space coordinate suffixes `x` and `y`.

The conservative flux source is defined as

$$\gamma = \begin{bmatrix} \begin{bmatrix} \gamma_{11} \\ \gamma_{12} \end{bmatrix} \\ \begin{bmatrix} \gamma_{21} \\ \gamma_{22} \end{bmatrix} \end{bmatrix}$$

Here the second index, j, of γ_{ij} denotes the space coordinate suffixes for `x` and `y`.

For the flux terms the divergence operator works on each row separately. To illustrate this, consider the divergence of the conservative flux source

$$\nabla \cdot \gamma = \nabla \cdot \begin{bmatrix} \begin{bmatrix} \gamma_{11} \\ \gamma_{12} \end{bmatrix} \\ \begin{bmatrix} \gamma_{21} \\ \gamma_{22} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \nabla \cdot \begin{bmatrix} \gamma_{11} \\ \gamma_{12} \end{bmatrix} \\ \nabla \cdot \begin{bmatrix} \gamma_{21} \\ \gamma_{22} \end{bmatrix} \end{bmatrix}$$

The convection term is defined as

$$\beta \cdot \nabla u = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \cdot \begin{bmatrix} \nabla u_1 \\ \nabla u_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \beta_{111} \\ \beta_{112} \end{bmatrix} & \begin{bmatrix} \beta_{121} \\ \beta_{122} \end{bmatrix} \\ \begin{bmatrix} \beta_{211} \\ \beta_{212} \end{bmatrix} & \begin{bmatrix} \beta_{221} \\ \beta_{222} \end{bmatrix} \end{bmatrix} \cdot \begin{bmatrix} \nabla u_1 \\ \nabla u_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \beta_{111} \\ \beta_{112} \end{bmatrix} \cdot \nabla u_1 + \begin{bmatrix} \beta_{121} \\ \beta_{122} \end{bmatrix} \cdot \nabla u_2 \\ \begin{bmatrix} \beta_{211} \\ \beta_{212} \end{bmatrix} \cdot \nabla u_1 + \begin{bmatrix} \beta_{221} \\ \beta_{222} \end{bmatrix} \cdot \nabla u_2 \end{bmatrix}$$

The variable names for these components are `beu1` and `beu2`.

The absorption term is defined as

$$au = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} a_{11}u_1 + a_{12}u_2 \\ a_{21}u_1 + a_{22}u_2 \end{bmatrix}$$

The variable names for these components are `au1` and `au2`.

The source term is defined as

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

The variable names for these components are `f1` and `f2`.

The Boundary Condition Terms

The Dirichlet boundary condition, in expanded form, reads

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

If you choose the Dirichlet condition, you also get the generalized Neumann boundary condition, which reads

$$\mathbf{n} \cdot (c\nabla u + \alpha u - \gamma) + qu = g - h^T\mu$$

The normal vector $\mathbf{n} = (n_x, n_y)$ operates on the flux vector in the same way as the divergence operator as explained earlier. If h has full rank (as in the default identity matrix, for example) only the constraints from the Dirichlet condition are active.

If you choose the Neumann condition, you get only the boundary condition

$$\mathbf{n} \cdot (c\nabla u + \alpha u - \gamma) + qu = g$$

The normal component of the diffusive flux is defined as

$$\mathbf{n} \cdot c\nabla u = \mathbf{n} \cdot \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \nabla u_1 \\ \nabla u_2 \end{bmatrix} = \begin{bmatrix} \mathbf{n} \cdot (c_{11}\nabla u_1 + c_{12}\nabla u_2) \\ \mathbf{n} \cdot (c_{21}\nabla u_1 + c_{22}\nabla u_2) \end{bmatrix}$$

The normal component of the conservative convective flux is defined as

$$\mathbf{n}\cdot\alpha u = \mathbf{n}\cdot\begin{bmatrix}\begin{bmatrix}\alpha_{111}\\\alpha_{112}\end{bmatrix}u_1+\begin{bmatrix}\alpha_{121}\\\alpha_{122}\end{bmatrix}u_2\\\begin{bmatrix}\alpha_{211}\\\alpha_{212}\end{bmatrix}u_1+\begin{bmatrix}\alpha_{221}\\\alpha_{222}\end{bmatrix}u_2\end{bmatrix} = \begin{bmatrix}(n_x, n_y)\cdot\left(\begin{bmatrix}\alpha_{111}\\\alpha_{112}\end{bmatrix}u_1+\begin{bmatrix}\alpha_{121}\\\alpha_{122}\end{bmatrix}u_2\right)\\(n_x, n_y)\cdot\left(\begin{bmatrix}\alpha_{211}\\\alpha_{212}\end{bmatrix}u_1+\begin{bmatrix}\alpha_{221}\\\alpha_{222}\end{bmatrix}u_2\right)\end{bmatrix}$$

The normal component of the conservative flux source is defined as

$$\mathbf{n}\cdot\gamma = (n_x, n_y)\cdot\begin{bmatrix}\begin{bmatrix}\gamma_{11}\\\gamma_{12}\end{bmatrix}\\\begin{bmatrix}\gamma_{21}\\\gamma_{22}\end{bmatrix}\end{bmatrix} = \begin{bmatrix}(n_x, n_y)\cdot\begin{bmatrix}\gamma_{11}\\\gamma_{12}\end{bmatrix}\\(n_x, n_y)\cdot\begin{bmatrix}\gamma_{21}\\\gamma_{22}\end{bmatrix}\end{bmatrix}$$

The boundary absorption term is defined as

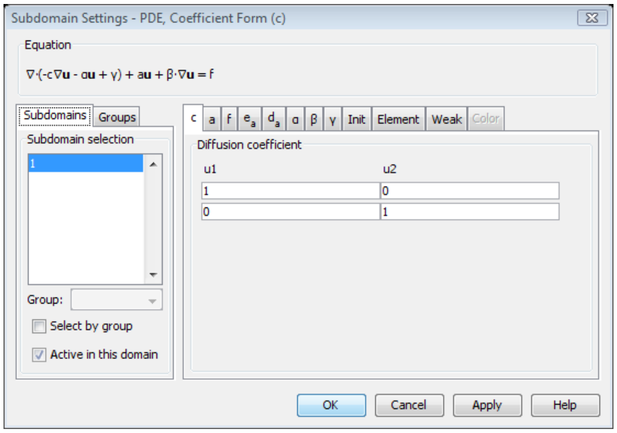
$$qu = \begin{bmatrix}q_{11} & q_{12}\\q_{21} & q_{22}\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} = \begin{bmatrix}q_{11}u_1+q_{12}u_2\\q_{21}u_1+q_{22}u_2\end{bmatrix}$$

The boundary source term is defined as

$$g = \begin{bmatrix}g_1\\g_2\end{bmatrix}$$

Specifying PDE Coefficients on Subdomains

The following figure shows the **Subdomain Settings** dialog box for a coefficient form, two-variable stationary 2D problem.


Subdomain Settings - PDE, Coefficient Form (c)

Equation

$\nabla\cdot(-c\nabla \mathbf{u} - \alpha\mathbf{u} + \gamma) + a\mathbf{u} + \beta\cdot\nabla\mathbf{u} = f$

Subdomains | Groups

Subdomain selection: 1

Group:

☐ Select by group

☑ Active in this domain

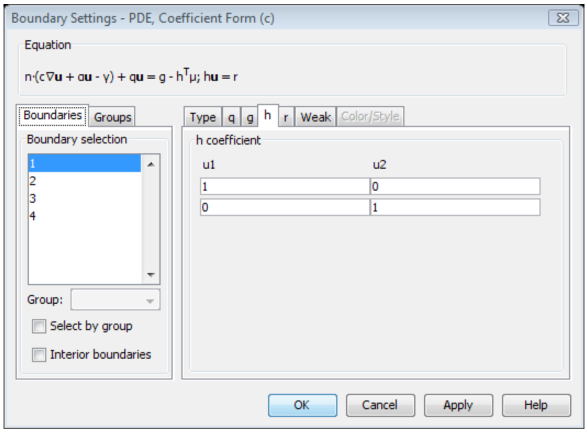
Tabs: c | a | f | e_a | d_a | α | β | γ | Init | Element | Weak | Color

Diffusion coefficient

u1	u2
1	0
0	1

Specifying Boundary Conditions

The **Boundary Settings** dialog box looks like this:


Boundary Settings - PDE, Coefficient Form (c)

Equation

$\mathbf{n}\cdot(c\nabla \mathbf{u} + \alpha\mathbf{u} - \gamma) + q\mathbf{u} = g - h^T\mu;\ h\mathbf{u} = r$

Boundaries | Groups

Boundary selection: 1, 2, 3, 4

Group:

☐ Select by group

☐ Interior boundaries

Tabs: Type | q | g | h | r | Weak | Color/Style

h coefficient

u1	u2
1	0
0	1

Using the General Form PDE

The general form PDE provides a framework for specification of PDEs that can be nonlinear.

THE SCALAR GENERAL FORM EQUATION

Use the *general form* for nonlinear PDEs. Assuming that you are working with a single dependent variable u, then a stationary problem in general form reads

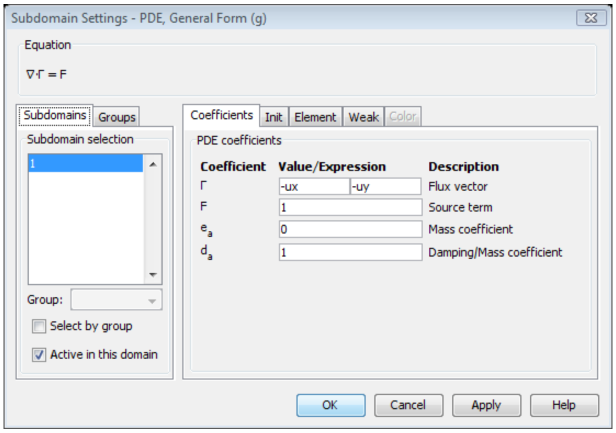
$$\begin{cases} \nabla \cdot \Gamma = F & \text{in } \Omega \\ -\mathbf{n} \cdot \Gamma = G + \left(\dfrac{\partial R}{\partial u}\right)^T \mu & \text{on } \partial\Omega \\ 0 = R & \text{on } \partial\Omega \end{cases}$$

The first equation is the PDE. The second and third equations are the Neumann and Dirichlet boundary conditions, respectively. For information on the time-dependent general form equation, see "Solving Time-Dependent Problems" on page 256.

The terms Γ, F, G, and R are coefficients. They can be functions of the spatial coordinates, the solution u, or the space derivatives of u. The coefficients F, G, and R are scalar, whereas Γ is the *flux vector*. The T in the Neumann boundary condition denotes the transpose. The variable μ is the Lagrange multiplier.

Specifying PDE Coefficients on Subdomains

The following image shows the **Subdomain Settings** dialog box for a general form, scalar, stationary 2D problem.



Subdomain Settings - PDE, General Form (g)

Equation

$\nabla \cdot \Gamma = F$

Subdomain selection: 1

Group:

Select by group

[✓ Active in this domain]

PDE coefficients

Coefficient	Value/Expression		Description
Γ	-ux	-uy	Flux vector
F	1		Source term
e_a	0		Mass coefficient
d_a	1		Damping/Mass coefficient

Specifying Boundary Conditions

The **Boundary Settings** dialog box entries are similar to the one for the coefficient form PDEs, but in general form you specify only the g and r coefficients.

THE GENERAL FORM EQUATION SYSTEM

In the case of several dependent variables $u_1, u_2, \dots, u_N$, the following system of equations represents a stationary problem in the general form:

$$\begin{cases} \nabla \cdot \Gamma_l = F_l & \text{in } \Omega \\ -\mathbf{n} \cdot \Gamma_l = G_l + \frac{\partial R_m}{\partial u_l}\mu_m & \text{on } \partial\Omega \\ 0 = R_m & \text{on } \partial\Omega \end{cases}$$

The equation index l ranges from 1 to N, while the constraint index m ranges from 1 to M. This discussion uses the summation convention. F_l, G_l, and R_m are scalars, whereas Γ_l is a vector. In this case there are several Lagrange multipliers: $\mu_1, \mu_2, \dots, \mu_M$.

For a more compact form, let u be a vector with components u_k, let Γ be a vector with components Γ_l, and so on. Then the system of equations takes on the same form as given above for a single dependent variable.

It is possible to rewrite the system to introduce the components Γ_{lj} of the vector Γ_l and the components n_j of the normal vector $\mathbf{n}$. Then the system of equations becomes

$$\begin{cases} \frac{\partial \Gamma_{lj}}{\partial x_j} = F_l & \text{in } \Omega \\ -n_j \Gamma_{lj} = G_l + \frac{\partial R_m}{\partial u_l}\mu_m & \text{on } \partial\Omega \\ 0 = R_m & \text{on } \partial\Omega \end{cases}$$

System for Two Variables in the General Form

The following example of a PDE in the general form is a stationary system for $N = 2$ solution components in $n = 2$ space dimensions with $M = 2$ constraints:

$$\begin{cases} \nabla \cdot \Gamma_1 = F_1 & \text{in } \Omega \\ \nabla \cdot \Gamma_2 = F_2 & \text{in } \Omega \end{cases}$$

with the generalized Neumann boundary conditions

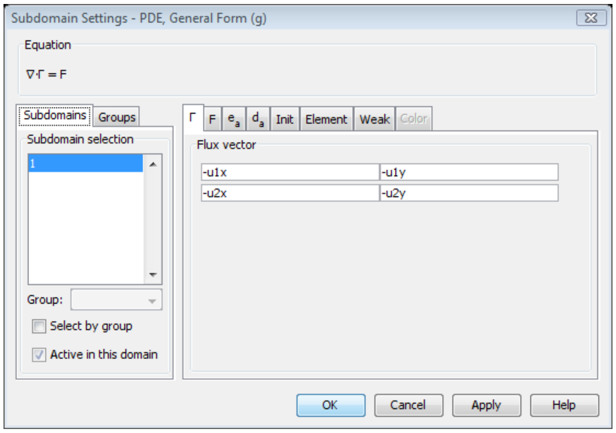
$$\begin{cases} -\mathbf{n}\cdot\Gamma_1 = G_1 + \dfrac{\partial R_1}{\partial u_1}\mu_1 + \dfrac{\partial R_2}{\partial u_1}\mu_2 & \text{on } \partial\Omega \\ -\mathbf{n}\cdot\Gamma_2 = G_2 + \dfrac{\partial R_1}{\partial u_2}\mu_1 + \dfrac{\partial R_2}{\partial u_2}\mu_2 & \text{on } \partial\Omega \end{cases}$$

and the Dirichlet boundary conditions

$$\begin{cases} 0 = R_1 & \text{on } \partial\Omega \\ 0 = R_2 & \text{on } \partial\Omega \end{cases}$$

Specifying PDE Coefficients on Subdomains

The following image shows the **Subdomain Settings** dialog box for a general form, 2-variable stationary 2D problem:



Subdomain Settings - PDE, General Form (g)

Equation

∇·Γ = F

Subdomain selection: 1

Group:

Select by group

Active in this domain: ✓

Flux vector

-u1x	-u1y
-u2x	-u2y

Specifying and Interpreting Boundary Conditions

The **Boundary Settings** dialog box is similar to the one for the coefficient form PDEs but in general form you specify only the *g* and *r* coefficients. To illustrate the flexibility of the boundary conditions, consider four cases:

- Let $R_1 = R_2 = 0$. Then the Dirichlet boundary conditions give $0 = 0$. In addition, the terms containing the Lagrange multipliers disappear from the Neumann boundary condition. Thus you have only the Neumann boundary conditions

$$\begin{cases} -\mathbf{n}\cdot\Gamma_1 = G_1 & \text{on } \partial\Omega \\ -\mathbf{n}\cdot\Gamma_2 = G_2 & \text{on } \partial\Omega \end{cases}$$

2 Let $R_1 = r_1 - u_1$ and $R_2 = r_2 - u_2$. Then the Dirichlet conditions are the usual $u_1 = r_1$ and $u_2 = r_2$. The Neumann boundary conditions become

$$\begin{cases} -\mathbf{n}\cdot\Gamma_1 = G_1 - \mu_1 & \text{on } \partial\Omega \\ -\mathbf{n}\cdot\Gamma_2 = G_2 - \mu_2 & \text{on } \partial\Omega \end{cases}$$

These equations impose no restrictions on u_1 or u_2, because the Lagrange multipliers μ_1 and μ_2 always adjust so as to fulfill the Dirichlet conditions. In this case, you can therefore ignore the Neumann boundary conditions.

3 Let $R_1 = r_1 - u_1$ and $R_2 = 0$. Then the Dirichlet conditions are

$$\begin{cases} 0 = r_1 - u_1 & \text{on } \partial\Omega \\ 0 = 0 & \text{on } \partial\Omega \end{cases}$$

and the Neumann conditions are

$$\begin{cases} -\mathbf{n}\cdot\Gamma_1 = G_1 - \mu_1 & \text{on } \partial\Omega \\ -\mathbf{n}\cdot\Gamma_2 = G_2 & \text{on } \partial\Omega \end{cases}$$

The first Neumann condition can be ignored because it imposes no restriction on u_1 or u_2. You effectively have only the Dirichlet condition on u_1 together with the second Neumann condition.

4 The same as case 3 above but with the two PDEs interchanged (Γ_1 and Γ_2 as well as F_1 and F_2). Then the PDEs are

$$\begin{cases} \nabla\cdot\Gamma_2 = F_2 & \text{in } \Omega \\ \nabla\cdot\Gamma_1 = F_1 & \text{in } \Omega \end{cases}$$

The Dirichlet condition is similar to that in Case 3: $u_1 = r_1$. The Neumann conditions then become

$$\begin{cases} -\mathbf{n}\cdot\Gamma_2 = G_2 - \mu_1 & \text{on } \partial\Omega \\ -\mathbf{n}\cdot\Gamma_1 = G_1 & \text{on } \partial\Omega \end{cases}$$

Effectively, you have only the Neumann condition $-\mathbf{n}\cdot\Gamma_1 = G_1$. In comparison with case 3 above, the PDEs and the Dirichlet conditions are identical, while the Neumann conditions are different. Thus, when mixing Dirichlet and Neumann conditions, the ordering of the equations and the dependent variables are important. However, the ordering of the Dirichlet conditions does not matter since the different Lagrange multipliers are for all practical purposes indistinguishable from each other.

5 Finally, let $R_1 = u_2 - u_1$ and $R_2 = 0$. Then the Dirichlet conditions are

$$\begin{cases} 0 = u_2 - u_1 & \text{on } \partial\Omega \\ 0 = 0 & \text{on } \partial\Omega \end{cases}$$

and the Neumann conditions are

$$\begin{cases} -\mathbf{n}\cdot\Gamma_1 = G_1 - \mu_1 & \text{on } \partial\Omega \\ -\mathbf{n}\cdot\Gamma_2 = G_2 + \mu_1 & \text{on } \partial\Omega \end{cases}$$

Note that the same Lagrange multiplier now appears in both Neumann conditions, which can have different definitions of Γ and G. Therefore, contrary to Cases 2 and 3 above, the Neumann conditions cannot be ignored. Instead, adding the two conditions, it becomes apparent that the solution and flux on the boundary must fulfill

$$\begin{cases} 0 = u_2 - u_1 & \text{on } \partial\Omega \\ -\mathbf{n}\cdot\Gamma_2 - \mathbf{n}\cdot\Gamma_2 = G_1 + G_2 & \text{on } \partial\Omega \end{cases}$$

In these examples, the values of the Lagrange multipliers do not matter. However, they often have a physical significance. In structural mechanics, the term

$$\left(\frac{\partial R}{\partial u}\right)^T \mu\,,$$

in the Neumann condition is the reaction force necessary to satisfy the kinematic constraints described by the Dirichlet conditions. This term has a special form because of the variational principles that give rise to it (see “Variational Principles” on page 312).

COEFFICIENT FORM VS. GENERAL FORM

The following substitutions show that the coefficient form and the general form are equivalent:

$$\Gamma = -c\nabla u - \alpha u + \gamma, \quad F = f - \beta\nabla u - au, \quad G = g - qu, \quad R = r - hu$$

Note: If r or h depend on u, there is a difference in the Neumann boundary condition because $\partial R/\partial u$ need not equal $-h$.

This duality lets you choose the representation that best suits a particular PDE. You can always convert a problem to a more general form:

- From the coefficient form to the general form
- From the coefficient or the general form to the weak form

Make this conversion by selecting another equation system form in the **Model Settings** dialog box.

COMSOL Multiphysics converts the coefficient form PDE to a general form PDE according to

$$\begin{aligned}
\Gamma_{lj} &= -c_{lkji}\frac{\partial u_k}{\partial x_i} - \alpha_{lkj}u_k + \gamma_{lj} \\
F_l &= f_l - \beta_{lki}\frac{\partial u_k}{\partial x_i} - a_{lk}u_k \\
G_l &= g_l - q_{lk}u_k \\
R_m &= r_m - h_{ml}u_l
\end{aligned}$$

using a notation where there is an implicit summation over the k (or l) and i indices in each product. The conversion only applies to the PDE coefficients and the boundary coefficients. Other settings on boundaries and subdomains such as initial values, shape functions, and weak form contributions remain unchanged.

Solving Time-Dependent Problems

To get the equation for a time-dependent PDE in COMSOL Multiphysics, add terms containing time derivatives to the left-hand side of the stationary equation. The time derivatives must appear linearly, and the Dirichlet conditions must be linear. A time-dependent problem in the coefficient form reads

$$\begin{cases} e_a\dfrac{\partial^2 u}{\partial t^2} + d_a\dfrac{\partial u}{\partial t} + \nabla\cdot(-c\nabla u - \alpha u + \gamma) + \beta\cdot\nabla u + au = f & \text{in } \Omega \\ \mathbf{n}\cdot(c\nabla u + \alpha u - \gamma) + qu = g - h^T\mu & \text{on } \partial\Omega \\ hu = r & \text{on } \partial\Omega \end{cases}$$

If u is a vector of dependent variables then the *mass coefficient* e_a is a matrix. All coefficients can depend on time. The name *mass matrix* or mass coefficient stems from the fact that in many physics applications e_a contains the mass density. The d_a coefficient represents damping for wave-type equations. However, if $e_a = 0$, then d_a is often called the mass coefficient. The default settings are $e_a = 0$ and $d_a = 1$, representing a time-dependent PDE such as the heat equation. Using $e_a = 1$ and $d_a = 0$ represents an undamped wave equation.

For a stationary model, COMSOL Multiphysics ignores any values or expressions that you enter in the edit field for the d_a and e_a coefficient. To activate the d_a and e_a coefficients and convert the model into a time-dependent model, select the time-dependent analysis type in the **Solver Parameters** dialog box.

The time-dependent problem in the general form is

$$\begin{cases} e_a\dfrac{\partial^2 u}{\partial t^2} + d_a\dfrac{\partial u}{\partial t} + \nabla\cdot\Gamma = F & \text{in } \Omega \\ -\mathbf{n}\cdot\Gamma = G + \left(\dfrac{\partial R}{\partial u}\right)^T\mu & \text{on } \partial\Omega \\ 0 = R & \text{on } \partial\Omega \end{cases}$$

For the weak form, the time-dependent problem (in 2D) looks like

$$\int_\Omega W^{(2t)}dA + \int_B W^{(1t)}ds + \sum_P W^{(0t)} = \int_\Omega W^{(2)}dA + \int_B W^{(1)}ds + \sum_P W^{(0)} +$$

$$+ \int_\Omega v_l \frac{\partial R_m^{(2)}}{\partial u_l} \mu_m^{(2)} dA + \int_B v_l \frac{\partial R_m^{(1)}}{\partial u_l} \mu_m^{(1)} ds + \sum_P v_l \frac{\partial R_m^{(0)}}{\partial u_l} \mu_m^{(0)}$$

$$0 = R^{(2)} \quad \text{on } \Omega$$

$$0 = R^{(1)} \quad \text{on B}$$

$$0 = R^{(0)} \quad \text{on P}$$

where the integrands $W^{(it)}$ depend bilinearly on the test functions v_l and on the first and second time derivatives of the dependent variables and their space derivatives. For example,

$$W^{(2t)} = v_1 \frac{\partial u_2}{\partial t} - \frac{\partial v_1}{\partial x} \cos(3u_1) \frac{\partial u_1}{\partial t} + v_2 \frac{\partial^2 u_2}{\partial t \partial y}$$

Time-Dependent Systems

For time-dependent systems the d_a and e_a coefficients are matrices. Assume you have the following system of equations

$$e_{11}\frac{\partial^2 u}{\partial t^2} + e_{12}\frac{\partial^2 u_2}{\partial t^2} + d_{11}\frac{\partial u_1}{\partial t} + d_{12}\frac{\partial u_2}{\partial t} + \nabla \cdot (-c_{11}\nabla u_1 - c_{12}\nabla u_2) + a_{11}u_1 + a_{12}u_2 = f_1$$

$$e_{21}\frac{\partial^2 u}{\partial t^2} + e_{22}\frac{\partial^2 u_2}{\partial t^2} + d_{21}\frac{\partial u_1}{\partial t} + d_{22}\frac{\partial u_2}{\partial t} + \nabla \cdot (-c_{21}\nabla u_1 - c_{22}\nabla u_2) + a_{21}u_1 + a_{22}u_2 = f_2$$

This system corresponds to the coefficient matrices

$$e_a = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}$$

$$d_a = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

$$c = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Many interesting problems have a singular d_a matrix (with $e_a = 0$), or a singular nonzero e_a matrix. Such problems are called *differential-algebraic equation* (DAE) systems. The COMSOL Multiphysics solver for time-dependent problems handles DAEs.

The models “Example: Resistive Heating” on page 334 and “Thermal Effects in Electronic Conductors” on page 36 in the *COMSOL Multiphysics Quick Start and Quick Reference* are both DAE systems.

Solving Eigenvalue Problems

COMSOL Multiphysics handles scalar eigenvalue problems for all PDE forms. These eigenvalue problems are related to time-dependent problems via the correspondence $\partial/\partial t \leftrightarrow -\lambda$, linking the time derivative to the eigenvalue λ (with the default eigenvalue name `lambda`). An eigenvalue problem in the coefficient form reads

$$\begin{cases} (\lambda-\lambda_0)^2 e_a u - (\lambda-\lambda_0) d_a u + \nabla\cdot(-c\nabla u - \alpha u) + \beta\cdot\nabla u + au = f & \text{in } \Omega \\ \mathbf{n}\cdot(c\nabla u + \alpha u) + qu = -h^T\mu & \text{on } \partial\Omega \\ hu = r & \text{on } \partial\Omega \end{cases}$$

where λ_0 is the linearization point for the eigenvalue. Note that the source terms are ignored if the solution form is coefficient form. If the general or weak solution forms are used, the source terms are not ignored if they depend on the solution components. If the coefficients depend on u or the eigenvalue λ, COMSOL Multiphysics performs a linearization of the problem about the linearization point $u = u_0$, $\lambda = \lambda_0$. The software also performs this linearization for eigenvalue problems in the general and weak forms, though in a slightly different way. See “The Linear or Linearized Model” on page 366 in the *COMSOL Multiphysics User’s Guide* for information about linearization.

To specify the linearization point u_0, use the settings in the **Values of variables not solved for and linearization point** area on the **Initial Value** page in the **Solver Manager** dialog box. To specify the linearization point λ_0 use the **Eigenvalue linearization point** edit field in the **Solver Parameters** dialog box. You can also change the name of the variable that should be treated as the eigenvalue in the **Eigenvalue name** edit field.

Interpreting PDE Coefficients

The COMSOL PDE formulations can model a variety of problems, but note that this documentation uses coefficient names that fall within the realm of continuum mechanics and mass transfer. For the coefficient form:

- e_a is the *mass coefficient*
- d_a is a *damping coefficient* or a *mass coefficient*.
- c is the *diffusion coefficient*.
- α is the *conservative flux convection coefficient*.
- β is the *convection coefficient*.
- a is the *absorption coefficient*.
- γ is the *conservative flux source term*.
- f is the *source term*.

$$\overset{\text{Mass}}{e_a\frac{\partial^2 u}{\partial t^2}} + \overset{\text{Damping/Mass}}{d_a\frac{\partial u}{\partial t}} - \nabla\cdot\underbrace{(\overset{\text{Diffusion}}{c\nabla u} + \overset{\text{Convection}}{\alpha u} - \overset{\text{Source}}{\gamma})}_{\text{Conservative Flux}} + \overset{\text{Convection}}{\beta\cdot\nabla u} + \underbrace{au}_{\text{Absorption}} = \overset{\text{Source}}{f}$$

There are many interesting PDE problems to which this interpretation does not apply. For instance, a time-harmonic PDE such as the Helmholtz equation represents a time-dependent phenomenon transformed into the frequency domain.

For the Neumann boundary condition of the coefficient form

$$\mathbf{n} \cdot (c\nabla u + \alpha u - \gamma) + qu = g - h^T\mu$$

- q is the *boundary absorption coefficient.*
- g is the *boundary source term.*

Classical PDEs

Many classical PDEs are instances of the coefficient form. All the classical PDEs in this section have their own application modes. To find them, open the **Model Navigator**. In the **Application Modes** tree, within the **COMSOL Multiphysics>PDE Modes** section, find the **Classical PDEs** folder. The table below shows the available classical PDEs using two notations: the compact notation of vector analysis (used in this documentation) and an expanded mathematical notation.

TABLE 9-2: CLASSICAL PDES IN COMPACT AND STANDARD NOTATION

EQUATION	COMPACT NOTATION	STANDARD NOTATION (2D)
Laplace's equation	$-\nabla \cdot (\nabla u) = 0$	$-\frac{\partial}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = 0$
Poisson's equation	$-\nabla \cdot (c\nabla u) = f$	$-\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right) = f$
Helmholtz equation	$-\nabla \cdot (c\nabla u) + au = f$	$-\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right) + au = f$
Heat equation	$d_a\frac{\partial u}{\partial t} - \nabla \cdot (c\nabla u) = f$	$d_a\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right) = f$
Wave equation	$e_a\frac{\partial^2 u}{\partial t^2} - \nabla \cdot (c\nabla u) = f$	$e_a\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right) = f$

TABLE 9-2: CLASSICAL PDES IN COMPACT AND STANDARD NOTATION

EQUATION	COMPACT NOTATION	STANDARD NOTATION (2D)
Schrödinger equation	$-\nabla\cdot(c\nabla u)+au = \lambda u$	$-\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right)+au = \lambda u$
Convection-diffusion equation	$d_a\frac{\partial u}{\partial t}-\nabla\cdot(c\nabla u)+\beta\cdot\nabla u = f$	$d_a\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right)+$ $\beta_x\frac{\partial u}{\partial x}+\beta_y\frac{\partial u}{\partial y}=f$

Equation Variables

The equation variables denote certain terms in the equations in coefficient form. Equation variables are available only when you have selected the coefficient or general solution form. Use the **Solution form** setting on the **Advanced** page in the **Solver Parameters** dialog box to set the solution form to **Coefficient** or **General**. Avoid using equation variables if there are other alternatives because you cannot use these variables when using the weak solution form. Often the application mode already contains variables with the corresponding quantities defined as application mode variables. The section "Using the Weak Solution Form" on page 294 in the *COMSOL Multiphysics Modeling Guide* explains why the weak solution form might be of advantage.

Recall the coefficient form system of equations

$$\begin{cases} e_{a\,lk}\frac{\partial^2 u_k}{\partial t^2}+d_{a\,lk}\frac{\partial u_k}{\partial t}+\nabla\cdot(-c_{lk}\nabla u_k-\alpha_{lk}u_k+\gamma_l)+\beta_{lk}\cdot\nabla u_k+a_{lk}u_k=f_l & \text{in } \Omega \\ \mathbf{n}\cdot(c_{lk}\nabla u_k+\alpha_{lk}u_k-\gamma_l)+q_{lk}u_k=g_l-h_{ml}\mu_m & \text{on } \partial\Omega \\ h_{mk}u_k=r_m & \text{on } \partial\Omega \end{cases}$$

where k and l range from 1 to N, and m ranges from 1 to M, where $M \le N$. Let y be the name of space coordinate number j. The notation in the following table uses the summation convention, that is, there is an implicit sum over all pairs of equal indices.

TABLE 9-3: EQUATION VARIABLES ON SUBDOMAINS

VARIABLE	MEANING	DESCRIPTION
`daul`	$d_{alk}u_k$	Mass coefficient multiplied with the solution
`culy`	$c_{lkji}\partial u_k/\partial x_i$	Component of the diffusive flux

TABLE 9-3: EQUATION VARIABLES ON SUBDOMAINS

VARIABLE	MEANING	DESCRIPTION
`alu`*`l`*`y`	$\alpha_{lkj}u_k$	Component of the conservative convective flux
`ga`*`l`*`y`	γ_{lj}	Component of the conservative flux source
`beu`*`l`*	$\beta_{lkj}\partial u_k/\partial x_j$	Convection term
`au`*`l`*	$a_{lk}u_k$	Absorption term
`f`*`l`*	f_l	Source term

TABLE 9-4: EQUATION VARIABLES ON BOUNDARIES

VARIABLE	MEANING	DESCRIPTION
`qu`*`l`*	$q_{lk}u_k$	Boundary absorption term
`g`*`l`*	g_l	Boundary source term

When there is only one `dim` variable, omit l from the equation variable names.

Note: In equation definitions, use the equation variables with caution to avoid circular definitions.

Boundary-Coupled Equation Variables

The boundary-coupled equation variables extrapolate the values of the equation variables to boundaries. The variables are available only when you have selected the coefficient or general solution form. To specify one of those solution forms, go to the **Solve** menu and open the **Solver Parameters** dialog box; click the **Advanced** tab and in the **Solution form** list select **Coefficient** or **General**. Avoid using equation variables if there are other alternatives because you cannot use these variables when using the weak solution form, which is the one in most cases when using the automatic solution form selection. Often the corresponding variables are already defined by the application mode as application mode variables. The section “Weak Form vs. Strong Forms” on page 295 in the *COMSOL Multiphysics Modeling Guide* explains why the weak solution form can be of advantage.

Recall the coefficient form system of equations

$$\begin{cases} e_{a\,lk}\dfrac{\partial^2 u_k}{\partial t^2} + d_{a\,lk}\dfrac{\partial u_k}{\partial t} + \nabla\cdot(-c_{lk}\nabla u_k - \alpha_{lk}u_k + \gamma_l) + \beta_{lk}\cdot\nabla u_k + a_{lk}u_k = f_l & \text{in } \Omega \\ \mathbf{n}\cdot(c_{lk}\nabla u_k + \alpha_{lk}u_k - \gamma_l) + q_{lk}u_k = g_l - h_{ml}\mu_m & \text{on } \partial\Omega \\ h_{mk}u_k = r_m & \text{on } \partial\Omega \end{cases}$$

The first type of boundary-coupled equation variables consists of equation variables restricted to the boundaries. Let l be an integer and y the name of space coordinate number j. The variables in the following table are defined on the boundaries. The upper and lower subdomains are the subdomains in the upward and downward directions (see the definition of normal vectors in "Direction of the Normal Component on Interior Boundaries" on page 167 in the *COMSOL Multiphysics User's Guide*). The table notation uses the summation convention, that is, there is an implicit summation over each pair of equal indices.

TABLE 9-5: BOUNDARY-COUPLED EQUATION VARIABLES

VARIABLE	MEANING
`ucu`*`l`*`y`	Value of $c_{lkji}\partial u_k/\partial x_i$ taken from the upper adjacent subdomain. 0 if there is no upper adjacent subdomain.
`dcu`*`l`*`y`	Value of $c_{lkji}\partial u_k/\partial x_i$ taken from the lower adjacent subdomain. 0 if there is no lower adjacent subdomain.
`cu`*`l`*`y`	If there are two adjacent subdomains, (`ucu`*`l`*`y`+`dcu`*`l`*`y`)/2. If there is one adjacent subdomain, the value $c_{lkji}\partial u_k/\partial x_i$ taken from this subdomain. Otherwise undefined.
`ualu`*`l`*`y`	Value of $\alpha_{lkj}u_k$ taken from the upper adjacent subdomain (0 if such does not exist)
`dalu`*`l`*`y`	Value of $\alpha_{lkj}u_k$ taken from the lower adjacent subdomain (0 if such does not exist)
`alu`*`l`*`y`	If there are two adjacent subdomains, (`ualu`*`l`*`y`+`dalu`*`l`*`y`)/2. If there is one adjacent subdomain, the value $\alpha_{lkj}u_k$ taken from this subdomain. Otherwise undefined.
`uga`*`l`*`y`	Value of γ_{lj} taken from the upper adjacent subdomain (0 if such does not exist)
`dga`*`l`*`y`	Value of γ_{lj} taken from the lower adjacent subdomain (0 if such does not exist)
`ga`*`l`*`y`	If there are two adjacent subdomains, (`uga`*`l`*`y`+`dga`*`l`*`y`)/2. If there is one adjacent subdomain, the value γ_{lj} taken from this subdomain. Otherwise undefined.

TABLE 9-5: BOUNDARY-COUPLED EQUATION VARIABLES

VARIABLE	MEANING
`ubeul`	Value of $\beta_{lkj}\partial u_k/\partial x_j$ taken from the upper adjacent subdomain (0 if such does not exist)
`dbeul`	Value of $\beta_{lkj}\partial u_k/\partial x_j$ taken from the lower adjacent subdomain (0 if such does not exist)
`beul`	If there are two adjacent subdomains, (`ubeul`+`dbeul`)/2. If there is one adjacent subdomain, the value taken from this subdomain. Otherwise undefined.
`uaul`	Value of $a_{lk}u_k$ taken from the upper adjacent subdomain (0 if such does not exist)
`daul`	Value of $a_{lk}u_k$ taken from the lower adjacent subdomain (0 if such does not exist)
`aul`	If there are two adjacent subdomains, (`uaul`+`daul`)/2. If there is one adjacent subdomain, the value a_{lk} taken from this subdomain. Otherwise undefined.
`ufl`	Value of f_l taken from the upper adjacent subdomain (0 if such does not exist)
`dfl`	Value of f_l taken from the lower adjacent subdomain (0 if such does not exist)
`fl`	If there are two adjacent subdomains, (`ufl`+`dfl`)/2. If there is one adjacent subdomain, the value f_l taken from this subdomain. Otherwise undefined.

When there is only one dependent variable, omit l from these variable names.

The second type of boundary-coupled equation variables involves the normal vector of the boundary. Some of these variables represent the jump in the terms in the normal component of the flux vector. Let n_j denote the components of the outward normal vector, seen from the upper or lower adjacent subdomain. The table notations use the summation convention.

TABLE 9-6: BOUNDARY COUPLED EQUATION VARIABLES INVOLVING NORMAL TO BOUNDARY

VARIABLE	MEANING
`uncul`	Value of $n_j c_{lkji}\partial u_k/\partial x_i$ taken from the upper adjacent subdomain. 0 if there is no upper adjacent subdomain.
`dncul`	Value of $n_j c_{lkji}\partial u_k/\partial x_i$ taken from the lower adjacent subdomain. 0 if there is no lower adjacent subdomain.
`ncul`	`uncul`+`dncul`. Since the normal vectors used in `uncul` and `dncul` are opposite, `ncul` is a jump in normal diffusive flux.

TABLE 9-6: BOUNDARY COUPLED EQUATION VARIABLES INVOLVING NORMAL TO BOUNDARY

VARIABLE	MEANING
`unalu`*l*	Value of $n_j\alpha_{lkj}u_k$ taken from the upper adjacent subdomain (0 if such does not exist)
`dnalu`*l*	Value of $n_j\alpha_{lkj}u_k$ taken from the lower adjacent subdomain (0 if such does not exist)
`nalu`*l*	`unalu`*l*`+dnalu`*l*
`unga`*l*	Value of $n_j\gamma_{lj}$ taken from the upper adjacent subdomain (0 if such does not exist)
`dnga`*l*	Value of $n_j\gamma_{lj}$ taken from the lower adjacent subdomain (0 if such does not exist)
`nga`*l*	`unga`*l*`+dnga`*l*

When there is only one dependent variable, omit l from the above variable names.

Note: In equation definitions, use boundary-coupled equation variables with caution to avoid circular definitions.

Ideal and non-Ideal Constraints

Constraints formulated through the coefficients h and r in the PDE, Coefficient Form application mode and through the coefficient R in the PDE, General Form application mode, give rise to constraints called *ideal.* An ideal constraint dictates exactly how the the flux conditions (or Neumann boundary conditions) are influenced by the constraint force. For the coefficent form, the flux condition is

$$\mathbf{n}\cdot(c\nabla u+\alpha u)+qu = g-h^T\mu,$$

and for the general form, the flux condition is

$$-\mathbf{n}\cdot\Gamma = G+\left(\frac{\partial R}{\partial u}\right)^T\mu \ .$$

The last term on the right-hand side in both expressions is the ideal constraint force. Thus, with ideal constraints you cannot enforce a flux condition independently of the constraints. In mathematics as well as in multiphysics modeling it is often necessary to enforce Neumann conditions and Dirichlet conditions more freely than what is possible through ideal constraints. As an example, consider again the general form example in Case 3 on page 253 and assume that you want to enforce the boundary conditions

$$\begin{cases} 0 = r_1 - u_1 & \text{on } \partial\Omega \\ -\mathbf{n} \cdot \Gamma_2 = G_2 & \text{on } \partial\Omega \end{cases}.$$

If $r_1 = r_1(u_2)$, the first condition is fulfilled but not the second. This is because the ideal constraint force is not zero:

$$-\mathbf{n} \cdot \Gamma_2 = G_2 + \frac{\partial R_1}{\partial u_2}\mu_1 + \frac{\partial R_2}{\partial u_2}\mu_2 = G_2 + \frac{\partial r_1}{\partial u_2}\mu_1 \neq G_2 .$$

To remedy this limitation with ideal constraints you can use *non-ideal constraints*, by which you can specify the constraint force independently of the constraints. The term non-ideal refers to the fact that the constraint force differs from $-h^T\mu$ for coefficient form and from $\left(\frac{\partial R}{\partial u}\right)^T \mu$ for general form. In multiphysics modeling, non-ideal constraints are, for example, necessary for the following boundary conditions:

- Normal-direction constraints on a moving mesh, where the mesh motion is part of the problem. These conditions are of the type $\mathbf{n}\cdot\mathbf{u} - r = 0$ where $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the boundary normal, **u** is a vector field (displacements or velocity), and **x** is the mesh coordinate vector. Ideal constraints give constraint forces not only on the equations for **u** but also on the equations for **x**, which typically are not wanted.
- Wall boundary conditions for turbulent fluid flow. For the k-ε turbulence model this condition is of the type $k - r(\varepsilon)$, $-n \cdot \nabla\varepsilon = 0$, where r is a given function. Ideal constraints for the first relation imply that the second relation cannot hold.

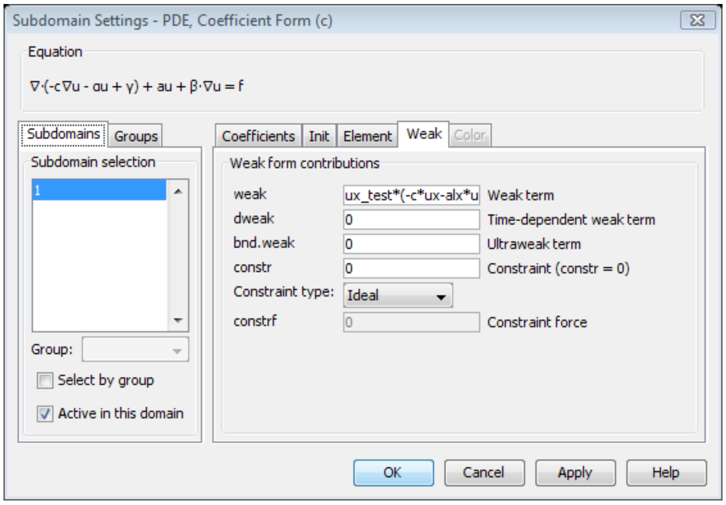
Non-ideal constraints can be enforced in COMSOL Multiphysics both in a pointwise and in a weak sense. For descriptions of how to use pointwise and weak non-ideal constraints see "Constraint Forces" on page 268 and "Specifying Weak Constraints" on page 301, respectively.

Variables for PDEs in Weak Form

When you use the weak solution form (the default solution form with the automatic solution form setting), the shape function variables (and the boundary-coupled shape variables) are the only automatically generated field variables available as defaults. The shape function variables are available in the same way as described earlier for the coefficient form and general solution form. User-defined expression variables and coupling variables are also available.

In addition to shape function variables, meta variables for test functions, test-function derivatives, and time derivatives are available in the **weak** and **dweak** edit fields on the **Weak** page in the **Subdomain Settings**, **Boundary Settings**, **Edge Settings**, and **Point Settings** dialog boxes. The test function meta variables follow a naming convention similar to that used for variables; see "Variable Naming Conventions" on page 164 in the *COMSOL Multiphysics User's Guide*.

To add weak terms, click the **Weak** tab in the **Point Settings**, **Edge Settings**, **Boundary Settings**, and **Subdomain Settings** dialog boxes. For the PDE, Coefficient Form application mode, the following dialog box is available for subdomains.



weak	ux_test*(-c*ux-alx*u	Weak term
dweak	0	Time-dependent weak term
bnd.weak	0	Ultraweak term
constr	0	Constraint (constr = 0)
Constraint type:	Ideal	
constrf	0	Constraint force

In the edit fields on the **Weak** page you can type a series of weak terms. The test functions always occur linearly in the weak terms. You can use these weak terms either alone to define the problem, as when using the weak form application modes, or in addition to the other coefficients in the coefficient form and general form application modes.

The following table shows the available test-function meta variables for a 2-variable weak form PDE. Let u be the name of the dependent variable and x, y, and z of the independent variables.

	1D	2D	3D
WEAK FORM META VARIABLE	`u_test`, `ux_test`	`u_test`, `ux_test`, `uy_test`	`u_test`, `ux_test`, `uy_test`, `uz_test`

The corresponding 2nd-order space derivatives are also available.

There is also a test operator with the same effect as the meta variables, so an alternative syntax to `u1_test`, for example, is `test(u1)`. See "Using Special Operators" on page 160 in the *COMSOL Multiphysics User's Guide* for more information about the test operator and other operators.

CONSTRAINT FORCES

On the **Weak** page you can also specify which constraint type to use. The default constraint type is *ideal*, which means that the PDE, Coefficient Form and PDE, General Form application modes derive the constraint force directly from the constraint specified in the **constr** field. Constraints specified using the **h** and **r** edit fields on the **Coefficients** page are always ideal and the **Constraint type** setting has no effect on them.

If you choose to use the default*non-ideal* constraints, the constraint force from a constraint written as **constr** = $R = 0$ on $\partial\Omega$, changes according to

$$\left(\frac{\partial R}{\partial u}\right)^T \mu \to -I_{\partial\Omega}\mu$$

where $I_{\partial\Omega}$ is an identity operator for points on the boundary $\partial\Omega$. If, for example, a two-variable problem for variables u_1 and u_2 specifies the constraints (using the **constr** field)

$$\begin{cases} 0 = R_1(u_1, u_2) & \text{on } \partial\Omega \\ 0 = R_2(u_1, u_2) & \text{on } \partial\Omega \end{cases},$$

then the default non-ideal constraint force is $-\mu_1$ acting on u_1 and $-\mu_2$ acting on u_2. The corresponding ideal constraint would give reaction forces

$$\left(\frac{\partial R_1}{\partial u_1}\right)^T \mu_1 + \left(\frac{\partial R_2}{\partial u_1}\right)^T \mu_2$$

acting on u_1 and

$$\left(\frac{\partial R_1}{\partial u_2}\right)^T \mu_1 + \left(\frac{\partial R_2}{\partial u_2}\right)^T \mu_2$$

acting on u_1.

If you want a constraint force different from any of these two choices, you can specify a separate expression for it. Select **User defined** from the **Constraint type** list and enter the expression in the **constf** edit field. In this edit field you specify the constraint force Jacobian using the test operator. For example, the expression corresponding to the default non-ideal constraint in the example above is

$$\begin{cases} \texttt{-test(u1)} \\ \texttt{-test(u2)} \end{cases}.$$

Note: The ideal constraint type normally generates a symmetric constraint Jacobian matrix and constraint force Jacobian matrix coupling for the discretized equations, while non-ideal constraints generate an unsymmetric coupling. See also "The Discretized Linearized Model" on page 367 in the *COMSOL Multiphysics User's Guide*.

Application Mode Variables

The following list shows the application mode variables that are available for postprocessing in addition to the other variables for PDEs:

NAME	TYPE	DESCRIPTION	EXPRESSION
u_i	S, B	The solution variable (dependent variable)	u_i
$u_i x_j$	S, B	The derivative of the solution variable u_i with respect to the space coordinate x_j, for example, uy	$\frac{\partial u_i}{\partial x_i}$

NAME	TYPE	DESCRIPTION	EXPRESSION
$u_ix_jx_k$	S, B	The second derivative of the solution variable u_i with respect to the space coordinates x_j and x_k, for example, uxx, uxy.	$\frac{\partial^2 u_i}{\partial x_j \partial x_k}$
u_it	S, B	The derivative of the solution variable u_i with respect to time	$\frac{\partial u_i}{\partial t}$
$u_i$tt	S, B	The second derivative of the solution variable u_i with respect to time	$\frac{\partial^2 u_i}{\partial t^2}$
u_ix_jt	S, B	The mixed derivative of the solution variable u_i with respect to time and the space coordinate x_j.	$\frac{\partial^2 u_i}{\partial x_j \partial t}$
absu_ix_i	S, B	Gradient of solution	$\lvert\nabla u\rvert, \quad \frac{\partial u_i}{\partial x_i}$
abscuix	S	Value of the c*grad(u_i) vector (Coefficient Form only). Only available when solving using coefficient form as solution form.	$\lvert c\nabla(u)\rvert$
absgaix	S	Value of the Gamma vector (General Form only). Only available when solving using general form as solution form.	$\lvert\Gamma\rvert$
cuix	S, B	c*grad(u_i) vector components (Coefficient Form only). Only available when solving using coefficient form as solution form.	$c\frac{\partial u_i}{\partial x}$
ga1xj	S, B	Gamma vector components (General Form only). Only available when solving using general form as solution form.	γ_x

NAME	TYPE	DESCRIPTION	EXPRESSION
ncu*i*	B	Normal components of the c*grad(u_i) vector (Coefficient Form only). Only available when solving using coefficient form as solution form.	$\mathbf{n}\cdot c\nabla u$
nga*i*	B	Normal components of the Gamma vector (General Form only). Only available when solving using general form as solution form.	$\mathbf{n}\cdot\gamma$

Note: All application mode variables include a suffix indicating which application mode they belong to. Table 9-1 on page 238 lists the default suffix for the PDE modes. For example, the default suffix added to application mode variable names for a General Form application mode is `_g`. The dependent variables u_i and their space and time derivatives do not include a suffix.

PDE Coefficients and Boundary Conditions with Time Derivatives

The d coefficients relate to 1st-order time derivatives of the solution u. The only directly available d coefficient is d_a, the coefficient in front of $\partial u/\partial t$; the subscript a is used because d_a is similar to the a coefficient in the absorption term except that it multiplies $\partial u/\partial t$ instead of u. In contrast, the coefficients d_c, $\mathbf{d}_\alpha$, and $\mathbf{d}_\beta$ for $\partial u/\partial t$—analogous to the coefficients c, α, and β for u, respectively—in the extended PDE formulation

$$-\nabla\cdot\left(d_c\nabla\frac{\partial u}{\partial t}+\mathbf{d}_\alpha\frac{\partial u}{\partial t}\right)+\mathbf{d}_\beta\cdot\nabla\frac{\partial u}{\partial t}+d_a\frac{\partial u}{\partial t}+\nabla\cdot(-c\nabla u+\dots)$$

are not directly available in the general or coefficient form PDE models. Instead, you can enter them in the existing γ and f terms:

- In 1D, add `-d_c*uxt-d_al*ut` to the γ term, and add `-d_be*uxt` to the f term.
- In 2D, add `-d_c*uxt-d_al1*ut` to the first γ component, and add `-d_c*uyt-d_al2*ut` to the second γ component. Add `-d_be1*uxt-d_be2*uyt` to the f term.

Replace the variables `d_c`, `d_al`, `d_be`, `d_al1`, `d_al2`, `d_be1`, and `d_be2` with appropriate expressions.

You enter 2nd-order time derivative terms in an analogous manner.

To specify a boundary condition containing time-derivative terms as in

$$\mathbf{n}\cdot(c\nabla u + \ldots) + e_q\frac{\partial^2 u}{\partial t^2} + d_q\frac{\partial u}{\partial t} + qu = g - h^T\mu\,,$$

add the terms `-e_q*utt-d_q*ut` to the g term, and provide appropriate values or expressions for the coefficients e_q and d_q in, for instance, the **Options>Expressions>Boundary Expressions** dialog box.

Using Equation Contributions on Interior Mesh Boundaries

In COMSOL Multiphysics it is possible to have equation contributions also from mesh element boundaries that does not lie on a geometry boundary. This makes additional finite element formulations possible like, for example, the discontinuous Galerkin method in space for some transport and flow problems, and the ultraweak variational formulation for wave problems driven by a single frequency.'

Note: The Acoustics Module provides a predefined interface for modeling pressure acoustics using the ultraweak variational formulation.

You specify these interior mesh boundary equation contributions on the corresponding subdomains. It is also possible to define expression variables to use on the interior mesh boundaries. Useful applications typically use discontinuous elements and the up and down operators in expressions to access values on elements adjacent to the interior mesh boundaries.

Specifying Equation Contributions

1. Open the **Subdomain Settings** dialog box from the **Physics** menu for any of the PDE Modes. For models with other application modes, choose **Equation System> Subdomain Settings**.
2. Select the subdomain where you want to enter an equation contribution on the interior mesh boundaries.
3. Click the **Weak** tab or the **bnd.weak** tab depending on the equation system form.
4. Enter your equation contributions in the **bnd.weak** edit field or the edit fields in the **bnd.weak** terms area.
5. Click the **Element** tab.
6. Enter the desired integration order for the ultraweak equation contributions in the **bnd.gporder** text field.
7. Click **OK**.

Specifying Interior Mesh Boundary Expressions.

To specify expression variables on the interior mesh boundaries, select **Interior Mesh Boundary Expressions** from the **Expressions** submenu on the **Options** menu.

The example in section “Upwinding Using Discontinuous Basis Functions” on page 112 in the *COMSOL Multiphysics Model Library* shows how to implement the discontinuous Galerkin method with upwinding stabilization using equation contributions on interior mesh boundaries.

Examples—Equation-Based Modeling

The models in this section solves Poisson's equation and compares the solution to the exact solution and implements a point source. They serve as introductions to equation-based modeling using PDEs in COMSOL Multiphysics. You find many more equation-based models in the chapter "Equation-Based Models" in the *COMSOL Multiphysics Model Library.*

Poisson's Equation on the Unit Disk

A classic PDE with a well-known behavior is Poisson's equation

$$\begin{cases} -\nabla \cdot (\nabla u) = f & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

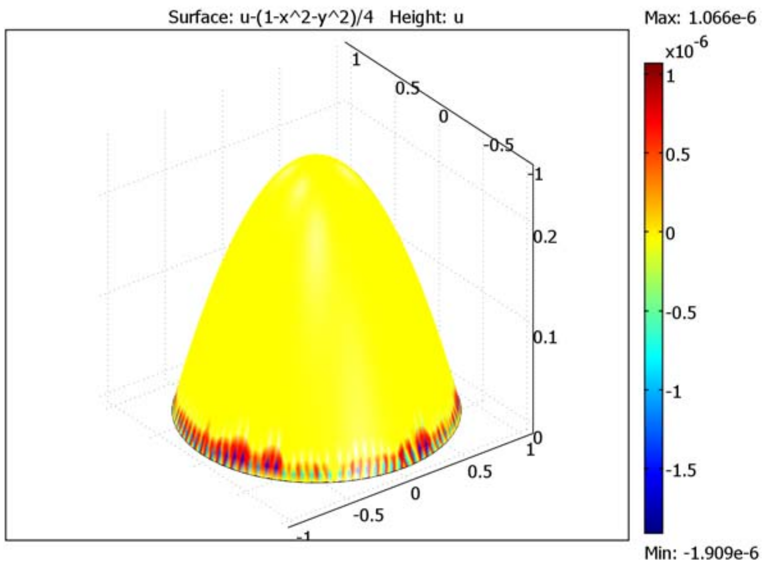
on the unit disk Ω with $f = 1$. The exact solution is

$$u(x, y) = \frac{1 - x^2 - y^2}{4},$$

which makes it possible to compare the COMSOL Multiphysics solution with the values of the exact solution at the node points on the mesh.

Results

The plot below shows the error (the difference between the numeric solution in COMSOL Multiphysics and the exact, analytic solution):



The finite element solution is very good near the center of the domain, which is natural considering that the exact solution and the element shape functions are both 2nd-order polynomials. Close to the boundary, however, the error becomes much larger. Because the element shape at the boundary follows the geometry, the relationship between local and global coordinates is not linear. Therefore, while being 2nd-order polynomials in their own local coordinates, the shape functions are not strictly 2nd-order polynomials in x and y.

Model Library path: `COMSOL_Multiphysics/Benchmark_Models/poisson_unit_disk`

Modeling Using the Graphical User Interface

The following section explains how to derive the above plot using the graphical user interface.

MODEL NAVIGATOR

1 In the **Model Navigator**, select **2D** from the **Space dimension** list.

2 In the list of application modes, open the **COMSOL Multiphysics>PDE Modes** folder and then the **Classical PDEs** folder.

3 Select **Poisson's Equation**.

4 Select **Lagrange - Quadratic** in the **Element** list (the default element type).

5 Click **OK**.

OPTIONS AND SETTINGS

1 Go to the **Options** menu and choose **Axes/Grid Settings**.

2 Enter the axis limits from the following table.

3 Click the **Grid** tab.

4 Clear the **Auto** check box and enter the following values for the grid spacings and extra grid points:

AXIS		GRID	
x min	`-2`	x spacing	`0.5`
x max	`2`	Extra x	
y min	`-1.5`	y spacing	`0.5`
y max	`1.5`	Extra y	

GEOMETRY MODELING

1 Find the Draw toolbar and click the **Ellipse/Circle (Centered)** button.

2 Using the right mouse button, draw a circle centered at (0, 0) with a radius of 1.

PHYSICS SETTINGS

Boundary Conditions

The default boundary condition in this mode is $u = 0$, so no changes are necessary.

Subdomain Settings

By default f is set to 1. Again, no changes are necessary.

MESH GENERATION

1 Click the **Mesh Mode** toolbar button to initialize and display the mesh.

2 Click the **Refine Mesh** toolbar button just once.

COMPUTING THE SOLUTION

Click the **Solve** toolbar button.

POSTPROCESSING AND VISUALIZATION

To view the solution as a 3D surface, click the **3D Surface Plot** button.

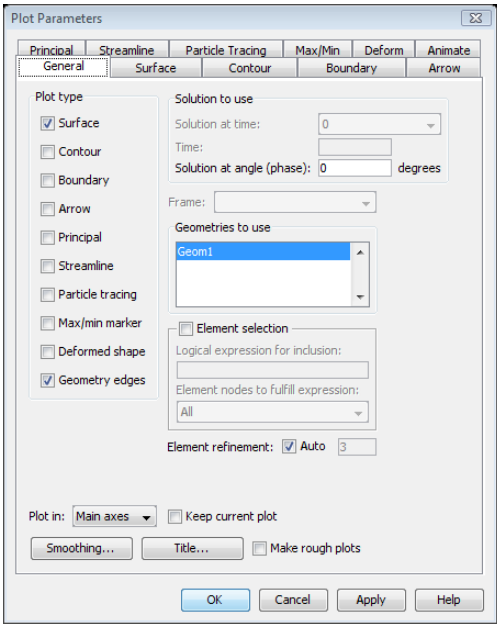
Compare the FEM solution to the exact solution:

1 Click the **Plot Parameters** toolbar button.

2 Click the **Surface** page.

3 On the **Surface Data** page, type `u-(1-x^2-y^2)/4` in the **Expression** edit field.

4 Click **OK**.



Plot type
- [✓] Surface
- [ ] Contour
- [ ] Boundary
- [ ] Arrow
- [ ] Principal
- [ ] Streamline
- [ ] Particle tracing
- [ ] Max/min marker
- [ ] Deformed shape
- [✓] Geometry edges

Solution to use
Solution at time: 0
Time:
Solution at angle (phase): 0 degrees

Frame:

Geometries to use
Geom1

[ ] Element selection
Logical expression for inclusion:
Element nodes to fulfill expression: All

Element refinement: [✓] Auto 3

Plot in: Main axes [ ] Keep current plot
[ ] Make rough plots

Implementing a Point Source

Consider Poisson's equation on the unit circle with a point source at the origin. Its formal expression is:

$$\begin{cases} -\nabla\cdot(\nabla u) = \delta & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

where δ is the Dirac δ distribution located at the origin. The exact solution is $u = -\frac{1}{2\pi}\log(r)$, which has a singularity at the origin.

The easiest way to describe a point source is with an extra weak term. Understanding why requires some basic knowledge of FEM theory and the weak formulation of PDEs. To obtain the weak formulation of the general Poisson equation (where f is any source term), multiply it with a test function u_{test} and integrate over the domain Ω:

$$\left\{\begin{array}{l} \int_\Omega -\nabla\cdot(\nabla u)u_{\mathrm{test}} = \int_\Omega f u_{\mathrm{test}} \\ \int_{\partial\Omega} u u_{\mathrm{test}} = 0 \end{array}\right., \forall u_{\mathrm{test}}$$

Integrate by parts and introduce the Lagrange multiplier λ to account for the constraint on the boundary. This results in the weak formulation:

$$\left\{\begin{array}{l} \int_\Omega \nabla u\cdot\nabla u_{\mathrm{test}} + \int_{\partial\Omega}\lambda u_{\mathrm{test}} = \int_\Omega f u_{\mathrm{test}} \\ \int_{\partial\Omega} u u_{\mathrm{test}} = 0 \end{array}\right., \forall u_{\mathrm{test}}$$

Most terms in these equations are sums of integrals over the domain, Ω, or over the domain boundary, $\partial\Omega$. You can explicitly add extra weak terms to the equations. In addition to handling terms integrated over Ω or $\partial\Omega$, COMSOL Multiphysics also handles contributions from integrals over edges and from single points. The software does not integrate contributions from points but instead collects them and adds them directly to the equations.

COMSOL Multiphysics discretizes the integral equations just given as long as the source term *f* is a function of the space coordinates, solution variables, and time. Unfortunately, you cannot express the Dirac δ distribution as a function that COMSOL Multiphysics can integrate numerically. The solution lies in noting that by the definition of the Dirac δ distribution, the following equality holds true:

$$\int_{\Omega} \delta u_{\mathrm{test}} = u_{\mathrm{test}}(0)$$

Therefore, setting *f* to zero and adding u_{test} as a weak term for a point at the origin provides the problem's correct weak formulation. The following model handles the resolution of the singularity at the origin using local mesh refinement.

Model Library path: `COMSOL_Multiphysics/Benchmark_Models/point_source`

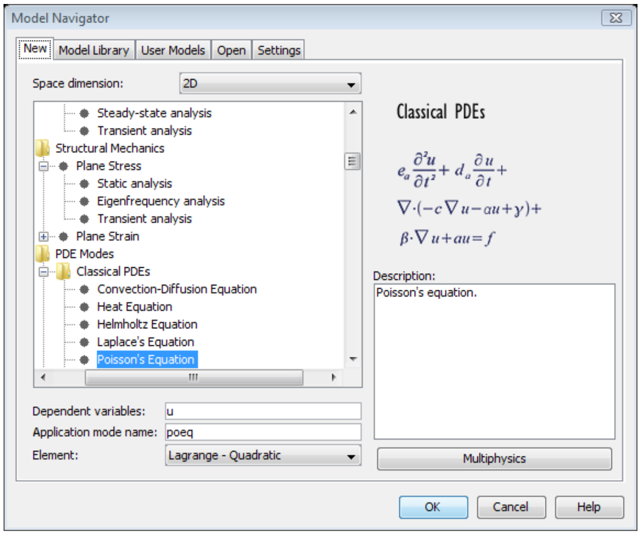
Modeling Using the Graphical User Interface

MODEL NAVIGATOR

1 In the **Model Navigator**, select **2D** in the **Space dimension** list.

2 In the list of application modes, open the **COMSOL Multiphysics>PDE Modes** folder and then **Classical PDEs**.

3 Select **Poisson's Equation**. In the **Element** list be sure to select **Lagrange - Quadratic**.



Space dimension: 2D

- Steady-state analysis
- Transient analysis
- Structural Mechanics
  - Plane Stress
    - Static analysis
    - Eigenfrequency analysis
    - Transient analysis
  - Plane Strain
- PDE Modes
  - Classical PDEs
    - Convection-Diffusion Equation
    - Heat Equation
    - Helmholtz Equation
    - Laplace's Equation
    - Poisson's Equation

Classical PDEs

$$e_a \frac{\partial^2 u}{\partial t^2} + d_a \frac{\partial u}{\partial t} +$$
$$\nabla\cdot(-c\nabla u - \alpha u + \gamma) +$$
$$\beta\cdot\nabla u + au = f$$

Description:
Poisson's equation.

Dependent variables: u
Application mode name: poeq
Element: Lagrange - Quadratic

4 Click **OK**.

OPTIONS AND SETTINGS

1 Go to the **Options** menu and choose **Axes/Grid Settings**.

2 In the **Axes/Grid Settings** dialog box, enter the following axis limits:

AXIS	
x min	-2
x max	2
y min	-1.5
y max	1.5

3 Click the **Grid** tab.

4 Clear the **Auto** check box.

5 Enter the following values in the edit fields for the grid spacings:

GRID	
x spacing	0.5
Extra x	
y spacing	0.5
Extra y	

GEOMETRY MODELING

1 Click the **Ellipse/Circle (Centered)** button on the Draw toolbar.

2 Using the right mouse button, draw a circle centered at (0, 0) with a radius of 1.

3 Click the **Point** button on the Draw toolbar and click once at the origin.

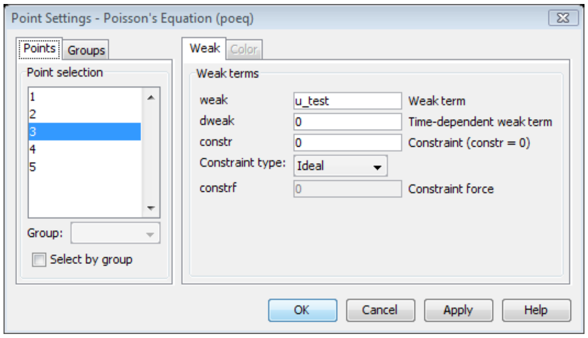
PHYSICS SETTINGS

Point Settings

1 Go to the **Physics** menu and choose **Point Settings**.

2 In the **Point Settings** dialog box select point **3**.

3 Type `u_test` in the **weak** edit field.


Point Settings - Poisson's Equation (poeq)

Point selection: 1, 2, 3 (selected), 4, 5

Group:

Select by group

Weak terms

weak	u_test	Weak term
dweak	0	Time-dependent weak term
constr	0	Constraint (constr = 0)
Constraint type:	Ideal	
constrf	0	Constraint force

4 Click **OK**.

Boundary Conditions

The default boundary condition in this mode is $u = 0$, so nothing must be changed.

Subdomain Settings

1 From the **Physics** menu choose **Subdomain Settings**.

2 Select Subdomain 1.

3 Enter these PDE coefficients; when finished, click **OK**.

PROPERTY	VALUE
c	1
f	0

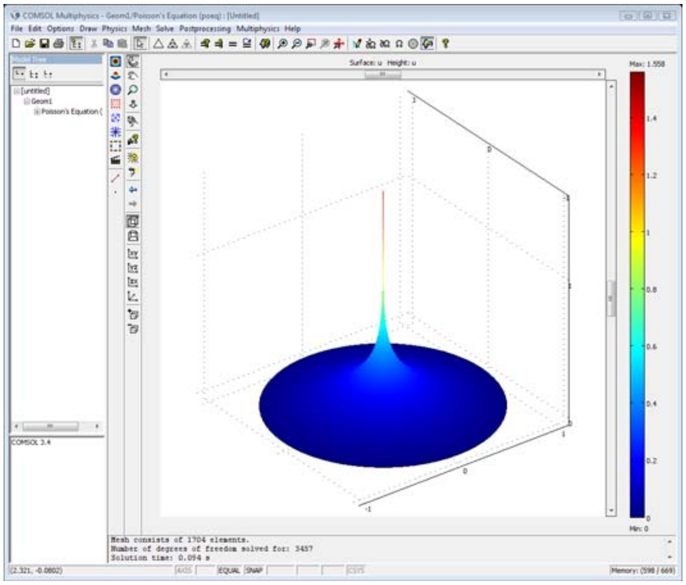
MESH GENERATION

Because the solution has a singularity at the origin, use a much higher mesh resolution around that point:

1 From the **Mesh** menu choose **Free Mesh Parameters**.
2 Click the **Point** tab.
3 Select Point 3.
4 In the **Maximum element size** edit field type `0.001`.
5 Click the **Remesh** button.

COMPUTING THE SOLUTION

Click the **Solve** button on the Main toolbar.


COMSOL Multiphysics - Geom1/Poisson's Equation (poeq) : [Untitled]
File Edit Options Draw Physics Mesh Solve Postprocessing Multiphysics Help
[untitled]
Geom1
Poisson's Equation (
COMSOL 3.4
Surface: u Height: u
Max: 1.558
Min: 0
(2.321, -0.0802)
EQUAL SNAP
Memory: (598 / 669)

```
Mesh consists of 1704 elements.
Number of degrees of freedom solved for: 3457
Solution time: 0.094 s
```

POSTPROCESSING AND VISUALIZATION

To view the solution as a 3D surface, click the **3D Surface Plot** button.

The exact solution is unbounded at the origin, which complicates evaluation of the error. Because the solution is axisymmetric, you can compare the COMSOL Multiphysics solution to the exact solution on a ray from just outside the origin to the boundary.

1 From the **Postprocessing** menu choose **Cross-Section Plot Parameters**.
2 In the **Cross-Section Plot Parameters** dialog box, click the **Line/Extrusion** tab.
3 In the **Expression** edit field under **y-axis data** type `u+log(x^2)/(4*pi)`.

4 In the **Cross-section line data** area, type `0.02` in the **x0** edit field and type `1` in the **x1** edit field. Leave both **y0** and **y1** at the default value of `0`.

5 Click **OK**.

Another way to check the solution's accuracy is to integrate the difference between the exact solution and the FEM solution over the domain. The result shows that although the local error close to the origin is quite large, the global error in the L^2 norm is small.

10

Global Equations and ODEs

This chapter describes the use of COMSOL Multiphysics to solve global equations such as ODEs, algebraic equations, and transcendental equations. You can solve such equations separately or coupled to a full finite-element model. Quick examples show how to specify and solve these types of equations.

Solving ODEs and Global Equations

Adding ODEs and Other Global Equations

When working on complex models, you frequently need to introduce single named degrees of freedom, or *states*, which are not logically related to any particular point in space, and their corresponding equations. In particular, such situations arise when modeling physics in interaction with an external system, for example, a controller or an electrical circuit built from standard components. It is often possible to describe such systems by a system of ordinary differential equations, ODEs, with a limited number of degrees of freedom.

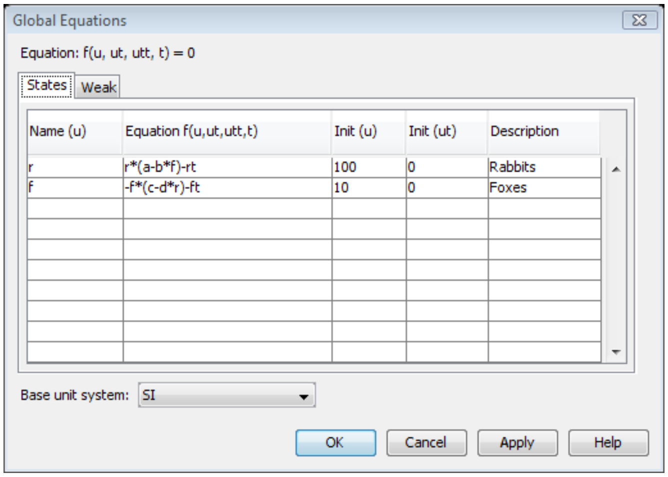
The **Global Equations** dialog box is designed for implementing this type of external equation, tightly coupled to a PDE model in a physical domain. You can use it for ODEs, differential algebraic equations, purely algebraic equations and conditions, and transcendental equations, or to add degrees of freedom to a PDE using the introduced states. Possible uses include:

- Controllers (see “Process Control Using a PID Controller” on page 317 in the *COMSOL Multiphysics Model Library*)
- Rigid-body mechanics (see “Terminal Falling Velocity of a Sand Grain” on page 237 in the *COMSOL Multiphysics Model Library*)
- Nonlinear eigenvalue problems (see “Example: Eigenmode Analysis of an L-Shaped Membrane with Rounded Corner” on page 425)
- Continuation
- Integral constraints (see “Example: Eigenmode Analysis of an L-Shaped Membrane with Rounded Corner” on page 425 of this book and “The Two-Term Boltzmann Equation” on page 171 in the *COMSOL Multiphysics Model Library*)
- Augmented or generalized equations (see “Stress-Optical Effects with Generalized Plane Strain” on page 615 in the *Structural Mechanics Module Model Library*, also available in the *RF Module Model Library*)

The Global Equations Dialog Box

To add a space-independent equation such as an ODE, choose **Global Equations** from the **Physics** menu. On the **States** page, each row corresponds to a named state, that is, a single degree of freedom. Fill in the name of the state variable in the first column

(**Name (u)**). This also defines time-derivative variables. If a state variable is called `u`, its first and second time derivatives are `ut` and `utt`, respectively. These variables become available in all domains in all geometries. Therefore the names must be unique.



Name (u)	Equation f(u,ut,utt,t)	Init (u)	Init (ut)	Description
r	r*(a-b*f)-rt	100	0	Rabbits
f	-f*(c-d*r)-ft	10	0	Foxes

Figure 10-1: The Global Equations dialog box, States page. The ODEs in the dialog box are the Lotka-Volterra or predator-prey equations.

Use the **Equation** column to specify an expression that is set equal to zero and added to the global system of equations. You can use all state variables and their time derivatives as well as any constants, global expression variables, and integration-coupling variables with a global destination. Setting an equation for a state is optional. The default value of `0` means that COMSOL Multiphysics does not add any additional condition to the model.

If the time derivative of a state variable appears somewhere in the model during a time-dependent solution, the state variable needs an initial condition. Models that contain second time derivatives also require an initial value for the first time derivatives of the state variables. You set these conditions in the third (**Init (u)**) and fourth (**Init (ut)**) columns. The last column, **Description**, gives you the option to enter comments about the state or the equation.

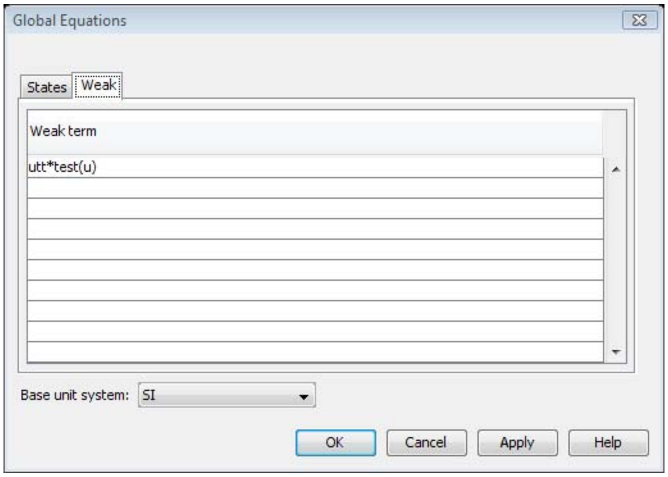
Solving ODEs—the Lotka-Volterra Equations

As an example, the ODEs in Figure 10-1 are the Lotka-Volterra equations:

$$\dot{r} = ar - brf$$
$$\dot{f} = -cf + drf$$

where r is the rabbit population, and f is the population of foxes.

Another option is to enter equations in the weak form on the **Weak** page. This is convenient in advanced modeling because it gives you control over the test variables multiplying the equations. Wherever a test function of a state variable appears (in the **Global Equations** dialog box or elsewhere in the model), whatever it multiplies ends up in the same equation in the discrete system. On the **Weak** page you can also use test functions of integration-coupling variables. This way you can keep the definition of a controller and its influence on a PDE in the same dialog box. Note that you can have zero or more weak expressions, regardless of the number of states.



Weak term
utt*test(u)

Base unit system: SI

Figure 10-2: The Global Equations dialog box, entering an ODE in the weak form.

To postprocess the solution to an ODE, type the name of the state variable in the **Expression** edit field in most postprocessing dialog boxes, for example, on the **Point** page in the **Domain Plot Parameters** dialog box for a plot of the time evolution. The state variables are available globally so you can select any point in the geometry.

Solving Algebraic and Transcendental Equations

As an example of an algebraic equation, consider the following:

$$f(u) = u^3 + u - 2$$

This equation has a single root at $u = 1$. To enter it into the **Global Equations** dialog box, type `u` in the **Name (u)** column and `u^3+u-2` in the **Equation** column (both entries on the same row). Then click the **Solve** button on the Main toolbar to compute the solution. To display the solution in the message log, choose **Postprocessing>Data Display>Global** and type `u` in the **Expression** edit field in the **Global Data Display** dialog box. Click **Apply**, and **Value: 1, Expression: u** appears in the message log.

As an example of a transcendental equation, consider the following:

$$f(u) = e^{-u} - u$$

A root to this equation is approximately 0.56714. To enter it into the **Global Equations** dialog box, type `u` in the **Name (u)** column and `exp(-u)-u` in the **Equation** column (both entries on the same row). Then click the **Solve** button on the Main toolbar to compute the solution. To display the solution in the message log, choose **Postprocessing>Data Display>Global** and type `u` in the **Expression** edit field in the **Global Data Display** dialog box. Click **Apply**, and **Value: 0.567143, Expression: u** appears in the message log.

Adding an ODE to a Boundary

Sometimes a boundary equation is coupled to a PDE in an adjacent subdomain. For example, adding an ODE for v on a boundary such that

$$\frac{\partial v}{\partial t} = u$$

where u is the dependent variable in an adjacent subdomain. To implement an ODE on a boundary, do the following:

1 Add a Weak Form, Boundary application mode.

2 Go to the **Physics** menu and choose **Boundary Settings**. In the **Boundary Settings** dialog box, click the **Weak** tab and type `v_test*u` in the **weak** edit field and `v_test*v_time` in the **dweak** edit field. This is the weak formulation of the ODE.

11

The Weak Form

This chapter explains weak form modeling, its purpose, and how to use it. The weak form in COMSOL Multiphysics is a particular way of specifying a model, one in which you use a more general syntax to describe the problem. The discussion concludes with a section giving some theoretical background.

Do not be misled by the term "weak:" the weak form is very powerful and flexible. We have borrowed the term *weak form* from mathematics, but in this context it has a slightly different meaning; this implementation incorporates features in addition to those defined in the mathematical weak form. Moreover, knowledge of the mathematical weak form is not prerequisite to using the COMSOL Multiphysics implementation.

The distinguishing features of the weak form in COMSOL Multiphysics are that it makes it possible to:

- Solve strongly nonlinear and nonlocal models that need an exact Jacobian for convergence.
- Enter certain equations which can be derived from an energy principle in a very compact and convenient form. Such equations, for example, arise in structural mechanics.

- Add and modify nonstandard constraints, such as various contact and friction models.
- Build models with PDEs on boundaries, edges, and points.
- Use the test operator to conveniently work with problems in variational calculus and parametric optimization.

COMSOL Multiphysics provides two different ways of modeling with the weak form:

- Weak form application modes
- Weak form contributions

In addition, you can add *weak constraints*, which, for example, provide accurate fluxes and reaction forces.

There are many additional benefits of the weak form, and we mention some of them in the context of specific models in other books in this documentation set.

Weak Form Modeling

The *weak form application modes* enable all of the weak form features outlined above. In particular, you need to use these application modes for models with separate PDEs on boundaries, edges, or points. However, modeling with the weak form does not always require using the weak form application modes; sometimes adding *weak form contributions* is sufficient, as described below.

By default, all models are converted to the *weak solution form* before solving. Although this feature is a part of the solution technique rather than part of the modeling process, it nonetheless belongs in this discussion because it is based on the weak form implementation. Using the *weak solution form* gives you the most important benefit of the weak form—the exact Jacobian necessary for fast convergence of strongly nonlinear problems—while you can still do the modeling and view the equations in the *equation form* of your choice.

The discussion in this chapter reviews the easiest ways to work with the weak form before going on to describe the weak form application modes.

Adding Weak Form Contributions

To add weak contributions to a model you are describing in a physics mode, follow these steps:

1. In the **Physics** menu, point to **Equation System** and then select **Subdomain Settings**, **Boundaries Settings**, **Edge Settings**, or **Points settings** depending on where you need to add the contributions.
2. In the corresponding dialog box, click the **Weak** tab.
3. Enter the weak contribution in the **weak** or **dweak** edit fields.

To add weak contributions to a model you are describing in a coefficient form or general form PDE mode, use the **Subdomain Settings** dialog box and other settings directly available in the **Physics** menu.

ADDING EDGE AND POINT SOURCES AND CONSTRAINTS

Using weak form contributions, you can also add sources and constraints on edges and points. The section “Specifying Point and Edge Settings” on page 242 in the *COMSOL Multiphysics User’s Guide* describes the procedure.

ADDING WEAK CONSTRAINTS

The weak form can assist in making accurate flux or reaction-force computations. The section "Using Weak Constraints" on page 300 describes how to use weak constraints for this purpose.

VARIATIONAL CALCULUS

Use the test operator to conveniently work with problems in variational calculus and optimization theory. See the minimal-surface example on page 296 for an example of this technique.

Modeling with PDEs on Boundaries, Edges, and Points

The weak form makes it possible to do PDE modeling on boundaries, edges, and points, thus opening up a wider class of models than the other application modes. This is the only type of modeling where you need the weak form application modes.

Using weak form application modes is also a smart way of handling thin layers; COMSOL Multiphysics then solves the problem by modeling rather than meshing. This approach reduces the solution time, and is sometimes what makes a solution possible at all.

The model "Transport and Adsorption" on page 30 in the *COMSOL Multiphysics Model Library* shows how to use the weak form boundary mode to model a thin adsorbtion layer with diffusion as a PDE on the boundary of a convection-diffusion problem. Another model using tangential derivative variables is "Shell Diffusion" on page 150 in the *COMSOL Multiphysics Model Library.*

Using Weak Coefficients or the Weak Modes

USING THE WEAK SOLUTION FORM

By using the weak solution form (the default solution form) you can model with one of the strong forms of the PDE using, for example, the PDE application modes, yet still take advantage of some of the abilities of the weak form. COMSOL Multiphysics converts all your equations to the weak form before solving the problem.

WEAK FORM VS. STRONG FORMS

Some features in COMSOL Multiphysics are available only in either strong or weak form. When a feature is available in the weak form but not in a strong form, you can always

- Display and modify the equations in the weak form using the weak equation system form
- Convert to the weak form when solving using the weak solution form (this is the default behavior)

When a feature is available only in a strong form, however, you must model using this form.

Strong Form Features

- When using residual method *Coefficient*, the adaptive mesh refinement considers only the strong form coefficients when computing the error estimators. The default residual method *Weak* accounts for all contributions, on weak and strong form.
- The equation variables and coupled boundary equation variables are available only for the strong forms.

Weak Form Features

The weak form automatically computes the Jacobian contributions for all variables, including coupling variables. The strong forms only consider Jacobian contributions from coefficients. Thus, to get an accurate Jacobian when dealing with for example coupling variables and mixed time-space derivatives you might have to use the weak solution form

Example: Variational Calculus

Variational calculus is a branch of mathematics where you want to find the function that extremizes some functional and often also determine the extremal value of the functional in question in the process. According to a fundamental result in the calculus of variations, the integral

$$\int_{\Omega} F(x, u, \nabla u) d\Omega$$

has a stationary value if the *Euler-Lagrange equation*

$$\frac{\partial F}{\partial u} - \nabla \cdot \frac{\partial F}{\partial \nabla u} = 0$$

is satisfied. Use the test operator to conveniently handle certain problems from variational calculus as it implements the Euler-Lagrange equation. See Chapter 4, "The Test Operator." in the *COMSOL Multiphysics User's Guide* for its definition.

Model Definition

The area of the surface $\{z = u(x,y) \mid (x,y) \in \Omega\}$ is equal to

$$A[u] = \int_{\Omega} \sqrt{1 + |\nabla u|^2} d\Omega$$

Given the values of the function u on the boundary $\partial\Omega$, the task is to determine u inside Ω such that the area is minimized. Solving this minimal-surface problem is related to finding the shape of a soap film with a prescribed boundary.

In this example, let $\Omega = \{(x, y) | x^2 + y^2 \le 1\}$ with $u = x^2$ on the boundary $\partial\Omega$. Using the Euler-Lagrange equation, you can derive the equation for the minimal-surface problem as:

$$-\nabla \cdot \left(\frac{1}{\sqrt{1 + |\nabla u|^2}} \nabla u \right) = 0$$

Simplify the problem by using the test operator to let COMSOL Multiphysics derive the minimal surface problem automatically. Typing in

$$\int_{\Omega} \text{test}(\sqrt{1+|\nabla u|^2})d\Omega$$

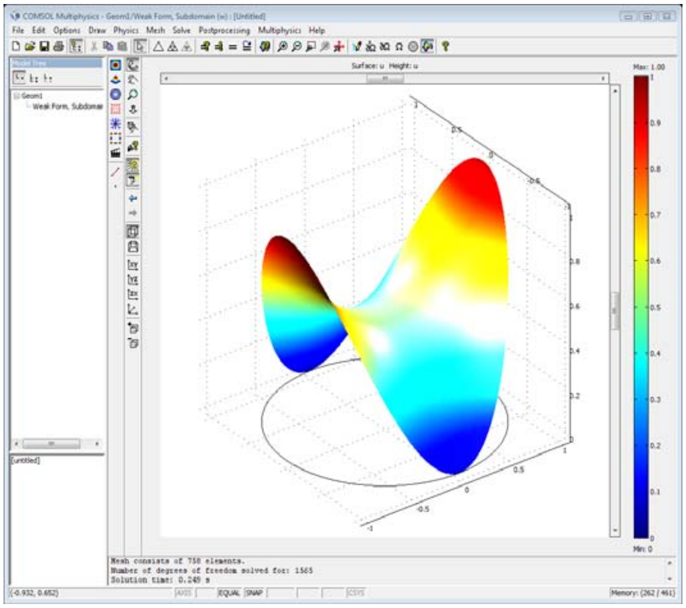
is equivalent to typing in

$$\int_{\Omega} \left(\text{test}(u)\frac{\partial}{\partial u}\sqrt{1+|\nabla u|^2} + \text{test}(\nabla u)\frac{\partial}{\partial \nabla u}\sqrt{1+|\nabla u|^2}\right)d\Omega$$

which is the weak form of the Euler-Lagrange equation.

Results

The shape of the solution illustrates how a soap film would be suspended in space, as constrained by the boundary.


COMSOL Multiphysics - Geom1/Weak Form, Subdomain (w) : [Untitled]
File Edit Options Draw Physics Mesh Solve Postprocessing Multiphysics Help
Model Tree
Geom1
Weak Form, Subdomain
[untitled]
Surface: u Height: u
Max: 1.00
Min: 0
(-0.932, 0.652)
AXIS
EQUAL
SNAP
CSYS
Memory: (262 / 461)

```
Mesh consists of 758 elements.
Number of degrees of freedom solved for: 1565
Solution time: 0.249 s
```

Modeling in COMSOL Multiphysics

Solve the problem directly in the weak form, using the stationary solver.

Model Library path: `COMSOL_Multiphysics/Benchmarks/minimal_surface`

Modeling Using the Graphical User Interface

MODEL NAVIGATOR

1 Select **2D** from the **Space dimension** list.

2 In the application mode list, open **COMSOL Multiphysics>PDE Modes** and then select **Weak Form, Subdomain**. Make sure **Lagrange - Quadratic** elements are selected.

3 Click **OK**.

GEOMETRY MODELING

The problem domain is a unit circle. Draw a circle centered at $(0, 0)$ with a radius of 1.

PHYSICS SETTINGS

Boundary Conditions

1 From the **Physics** menu, choose **Boundary Settings**.

2 Enter the following boundary coefficients:

SETTINGS	ALL BOUNDARIES
weak	`0`
dweak	`0`
constr	`x^2-u`

3 Click **OK**.

Subdomain Settings

1 From the **Physics** menu, open the **Subdomain Settings** dialog box and enter the PDE coefficients:

SETTINGS	SUBDOMAIN 1
weak	`test(sqrt(1+ux^2+uy^2))`
dweak	`0`
constr	`0`

2 Click **OK**.

MESH GENERATION

Click the **Initialize Mesh** button to create a mesh.

COMPUTING THE SOLUTION

Click the **Solve** button on the Main toolbar to solve the problem.

POSTPROCESSING AND VISUALIZATION

Click the **3D Surface Plot** button and the **Scene Light** button to create an illuminated 3D plot.

Modeling Using the Programming Language

Run the model from COMSOL Script or MATLAB:

1 Create the geometry and a standard mesh:

```
clear fem
fem.geom = circ2(0,0,1);
fem.mesh = meshinit(fem);
```

2 Set up the minimization problem that characterizes the minimal surface. Use second order Lagrange elements:

```
fem.shape = 2;
fem.sshape = 2;
fem.bnd.constr = 'x^2-u';
fem.equ.weak = 'test(sqrt(1+ux^2+uy^2))';
```

3 Create the extended mesh, solve the nonlinear problem, and plot the solution:

```
fem.xmesh = meshextend(fem);
fem.sol = femstatic(fem);
postsurf(fem,'u','u');
```

4 Compute the area of the minimal surface:

```
postint(fem,'sqrt(1+ux^2+uy^2)')
```

Using Weak Constraints

The *weak constraints* feature in COMSOL Multiphysics implements constraints by using finite elements on the constrained domain for the Lagrange multipliers, and by solving for the Lagrange multipliers along with the original problem. The weak constraints have a number of distinct advantages. They can:

- Provide very accurate flux computations—the Lagrange multipliers along the constraints optimally balance the finite element projection of the applied loads (reaction forces). See "Computing Accurate Fluxes" on page 253 in the *COMSOL Multiphysics User's Guide* for more information and an example.
- Handle nonlinear constraints—the nonlinear solver does not store the Lagrange multipliers arising from standard constraints from one step to the next, which affects the convergence when constraints are nonlinear. Weak constraints can handle general nonlinearities because the Lagrange multipliers are updated as a part of the solution vector and give correct contributions to the stiffness matrix.
- Implement constraints including derivatives—constraints only on the tangential component of the derivative work when using standard constraints, whereas you must use weak constraint to be able to handle nontangential constraints. Note that boundary conditions involving the normal component of the derivative can almost always be reformulated as a Neumann condition, which is usually preferable.

Weak constraints also come with the following drawbacks:

- Because extra unknowns are introduced for the Lagrange multipliers the problem size increases compared to when eliminating the constraints.
- The formulation normally implies that a so-called saddle-point problem is introduced with zeros on the main diagonal of the Jacobian of the discretized equations. This class of problems is often more difficult to solve. Because iterative methods for linear systems are sensitive to the eigenvalue distribution of the system matrix, a weak constraint formulation can be much more difficult to solve than the corresponding pointwise constraint formulation.
- Discontinuous constraints result in (theoretically) infinite Lagrange multipliers. In practice you get large oscillations.

Specifying Weak Constraints

In the graphical user interface you can add weak constraints using the **Weak constraints** application mode property. The following procedure applies to the physics modes:

1 From the **Physics** menu select **Properties**.

2 In the **Application Mode Properties** dialog box, select **On** from the **Weak constraints** list.

3 Select **Ideal** or **Non-ideal** from the **Constraint type** list (see "Ideal vs. Non-Ideal Constraints" on page 301)

4 Click **OK**.

The variable names in the application modes using weak constraints, such as `lm1`, correspond to the Lagrange multipliers. The weak constraints supersede the original constraints and implement them using a weak formulation. By default, weak constraints are active on all boundaries. To control the use of weak constraints on individual boundaries, open the **Boundary Settings** dialog box and click the **Weak Constr.** tab. On this tab, which appears when you add weak constraints, you can select boundaries and then select or clear the **Use weak constraints** check box to include or turn off the weak constraints for the selected boundaries. You can also set element type, shape function, and initial values for the weak constraint variables (Lagrange multipliers).

IDEAL VS. NON-IDEAL CONSTRAINTS

Ideal weak constraints in theory implement the same boundary conditions as the standard, pointwise, constraints. In some cases, the result may differ slightly, but this is only due to the different discretization. Using non-ideal constraints, on the other hand, modifies the way boundary conditions are interpreted. Switching from pointwise constraints to non-ideal weak constraints therefore can modify the *physics* of the model, while using ideal weak constraints only affects the *numerics*.

The difference between ideal and non-ideal constraints is the way the Lagrange multipliers—which can be interpreted as generalized reaction forces—are applied. In an ideal constraint, the Lagrange multipliers are applied symmetrically on all dependent variables involved in the constraint so as to keep symmetric problems symmetric. In a non-ideal constraint, the reaction forces are applied only on the application mode's dependent variable at the boundary where the constraint is specified.

Imagine a model with two application modes A and B, with dependent variables u and v, respectively. If application mode A specifies a standard constraint as $R = 2u - 3v = 0$, COMSOL Multiphysics implements the ideal weak boundary constraint by adding the weak term

$$\int_B \hat{\mu} \cdot (2u - 3v)ds + \int_B \mu \cdot (2\hat{u} - 3\hat{v})ds$$

where μ and $\hat{\mu}$ are the Lagrange multiplier and corresponding test function, and $\hat{u}$ and $\hat{v}$ are the test functions corresponding to u and v, respectively. Since μ multiplies both test functions in the second integral, both u and v are affected by the constraint, which is therefore *bidirectional.*

The corresponding non-ideal weak constraint adds the weak form contribution

$$\int_B \hat{\mu} \cdot (2u - 3v)ds + \int_B \mu \cdot \hat{u}ds$$

Here, the Lagrange multiplier μ only multiplies the test function $\hat{u}$. All reaction forces therefore apply only to u, while v is left unaffected by the constraint. This makes the non-ideal constraint *unidirectional.* Note also that the non-ideal constraint does not consider the factor 2 in front of u when applying the forces. Even if the constraint would contain spatial derivatives, the Lagrange multiplier would still be applied only directly on the dependent variable itself.

For more information on the effects of ideal and non-ideal constraints, see "Example—Coupling Variables and Boundary Constraints" on page 304. For a model that uses ideal weak constraints for accurate outward normal current densities, see "Semiconductor Diode" on page 442 in the *COMSOL Multiphysics Model Library.* For a model that uses non-ideal weak constraints to get the correct reaction forces, see "Sloshing Tank" on page 247 in the *COMSOL Multiphysics Model Library.*

Weak Constraints in Assemblies

When using assemblies, the application modes add a constraint on the dependent variables on the destination domains of the identity pairs. If you use weak constraints, the application modes also turn these constraints into weak constraints. Unlike other physics settings on pairs you do not control the use of weak constraints on pairs on the **Pair** tab in the **Boundary Settings** dialog box. If you would like to change the settings for the weak constraints on the destination boundaries of a pair go to the **Boundaries** tab and change the settings there. See the section "Specifying Physics Settings on Pairs" on page 361 for general information about physics settings on pairs.

The constraints on pairs, which makes sure that the dependent variables are continuous across the pair boundary, are always implemented as ideal constraints. This because the non-ideal constraints do not conserve the flux from the source side to the destination side of the pair.

Limitations of Weak Constraints

Weak constraints are more difficult to use than conventional pointwise constraints. Consider the following tips when using them:

- Pointwise and weak constraints on the same set of variables on adjacent boundaries (boundaries that share common node points in the mesh) do not work. This means that if you must constrain all boundaries on a solid and want to use a weak constraint on one boundary segment (one face), you must use the weak constraint on the entire boundary of the solid (if the boundary is connected).
- You must always have a nontrivial constraint (that is, a Dirichlet condition) on the boundaries where you enable the weak constraint for the constraint to take effect.
- Problems with the scaling of linear systems sometimes arise in conjunction with weak constraints. The Lagrange multiplier variables always have a different unit than the main system variables, and can therefore be of a completely different order of magnitude. Usually the automatic variable scaling in the solvers is sufficient, but there are cases when you may want to consider manual scaling.
- Always use the same shape function type for the weak constraint as for the variables that you constrain, possibly with lower-order elements for the weak constraint. In some cases, for example when constraining derivatives, the system of equations may become singular. The reason is usually that there are redundant Lagrange multiplier degrees of freedom in the model. Try to lower the order of the Lagrange multiplier variables, or use constraints on the Lagrange multiplier to remove some degrees of freedom.
- If you want to use iterative methods for the linear system and if the weak constraint formulation gives zero's on the diagonal of the system matrix, the SOR class of preconditioners and smoothers will not work. Instead try use the Vanka algorithm with the Lagrange multipliers as the Vanka variables, or use the incomplete LU factorization algorithm.

Example—Coupling Variables and Boundary Constraints

Boundary Constraints in a Heat Transfer Model

This example describes the difference between different boundary constraints when using a coupling variable to map a function of a solution from one boundary to another.

Consider the following example:

MODEL NAVIGATOR

1 In the **Model Navigator**, select **2D** in the **Space Dimension** list.

2 In the list of application modes, select **COMSOL Multiphsyics>Heat Transfer>Conduction>Steady-state analysis**.

GEOMETRY MODELING

1 Draw a rectangle with diagonal corners at (0, 0) and (2, 1).

2 Draw a circle with center at (1, 0.5) and radius 0.1.

3 Subtract the circle from the rectangle to create a hole: Press Ctrl+A to select both objects and click the **Difference** button.

PHYSICS SETTINGS

Boundary Conditions

1 From the **Physics** menu, open the **Boundary Settings** dialog box.

2 Select the boundaries of the circle (Boundaries 5–8) and select **Temperature** in the **Boundary condition** list. Type `300` in the **Temperature** edit field to specify the value of the fixed temperature (in kelvin). The circle then acts as a heat sink.

3 Select boundary 3 (the top boundary) and then select **Heat flux** in the **Boundary condition** list. Type `10000*x` in the **Inward heat flux** edit field. This creates a heat flux of 0 to 20,000 W/m^2, which increases linearly from left to right.

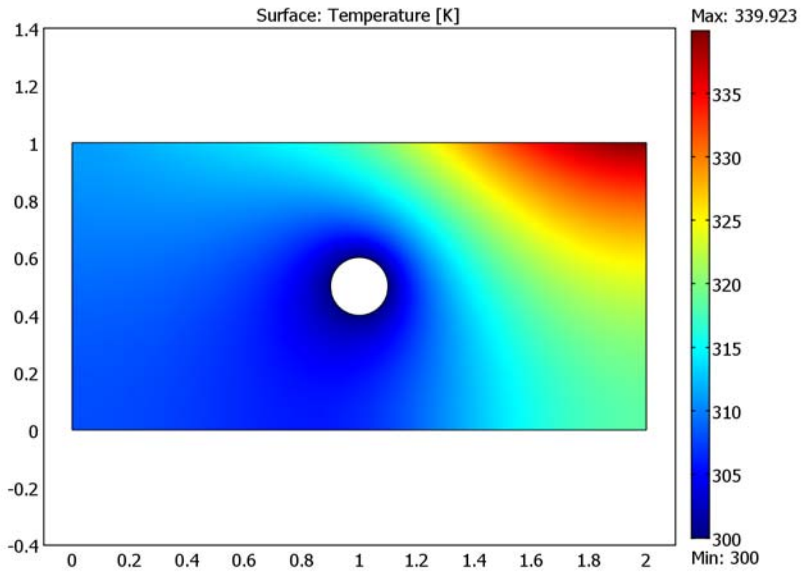
4 The other boundaries are thermally insulated, which is the default setting. Click **OK** to close the **Boundary Settings** dialog box.

Subdomain Settings

There are no changes to the settings in the **Subdomain Settings** dialog box (the default material properties are for copper).

COMPUTING THE SOLUTION

Click the **Solve** button (**=**) on the Main toolbar (or select **Solve Problem** from the **Solve** menu) to solve the problem.



The maximum temperature, about 340 K, appears in the upper-right corner.

Now, what if you want to set the temperature along the left boundary of the rectangle equal to the average temperature of the right boundary? To do so, you can use an integration coupling variable, which computes the average temperature of the right boundary and stores the value in a variable.

OPTIONS AND SETTINGS

1 From the **Options** menu, choose **Integration Coupling Variables>Boundary Variables**.

2 Select boundary 4 (the rightmost boundary). Define an integration coupling variable `Tint` for the integrated temperature in the top row by typing `Tint` in the **Name** column and `T` in the **Expression** column. Also, define a second variable on the next row with the name `blen` and an expression (integrand) that is `1`. The variable

`Tint` stores the value of the integral of T along the boundary, and the variable `blen` stores the value of the length of the boundary. The average temperature then becomes `Tint/blen`.

3 Click **OK** to close the **Boundary Integration Variables** dialog box.

BOUNDARY CONDITIONS

4 From the **Physics** menu, open **Boundary Settings**.

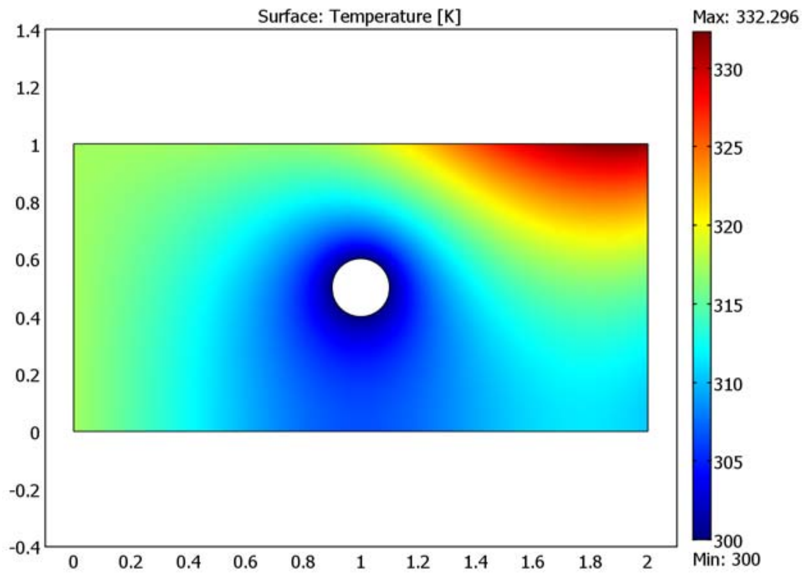
5 Select Boundary 1 (the leftmost boundary) and then select **Temperature** from the **Boundary condition** list.

6 Type `Tint/blen` in the **Temperature** edit field.

7 Click **OK** to close the **Boundary Settings** dialog box.

COMPUTING THE SOLUTION

Click the **Solve** button on the Main toolbar.



The maximum temperature is about 332 K close to, but not exactly in, the upper-right corner. The default type of boundary constraint in COMSOL Multiphysics is an *ideal boundary constraint*, which is a bidirectional constraint. COMSOL Multiphysics supports two different kinds of ideal boundary constraints: pointwise and weak. Now switch to a unidirectional, or non-ideal, boundary constraint, and solve the modified problem.

PHYSICS SETTINGS

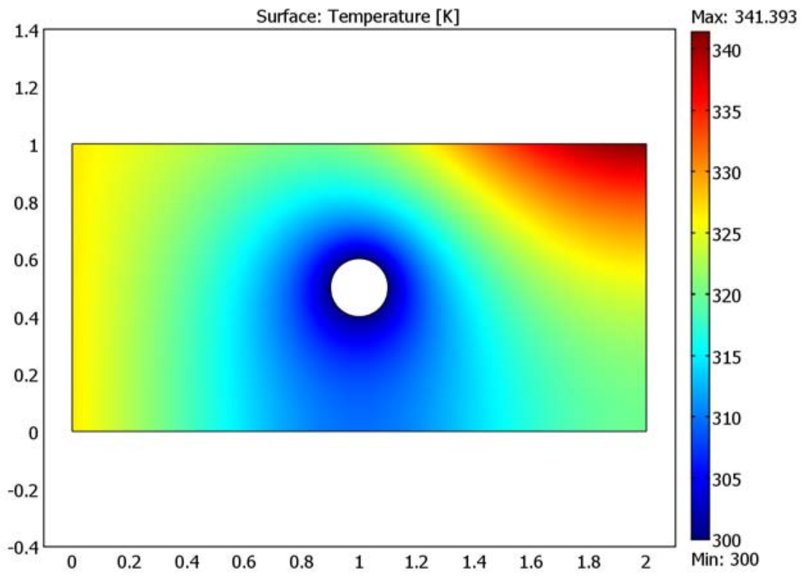
1 From the **Physics** menu, select **Properties**.

2 Select **On** from the **Weak constraints** list and **Non-ideal** from the **Constraint type** list to change from no weak constraints (equal to pointwise ideal constraint) to non-ideal weak constraints.

3 Click **OK** to close the **Application Mode Properties** dialog box.

COMPUTING THE SOLUTION

Click the **Solve** button on the Main toolbar to solve the problem with the modified constraints.



This time the maximum temperature is about 341 K, in the exact upper-right corner.

POSTPROCESSING

Do the following steps if you want to check that this is the average temperature:

1 From the **Postprocessing** menu, select **Data Display>Global**.

2 For the **Expression to evaluate**, type `Tint/blen`. Click **OK**.

This displays the average temperature value, about 327 K, in the message log. To compare this with the values in the plot, just click close to the left boundary. This displays the solution at the grid locations of the background grid, which you can see in the Draw mode. The same value appears for all points on the left boundary.

The subsequent sections explain the differences between the different types of boundary constraints.

No Weak Constraints

The pointwise ideal boundary constraint is what you get if you select **Off** from the **Weak constraints** list and **Ideal** from the **Constraint type** list. These are the default settings and correspond to the most common type of boundary constraint for virtually all boundary conditions of the Dirichlet type. In the previous example, when using the pointwise ideal boundary constraint together with the integrated coupling variables, the coupling is bidirectional.

Weak Ideal Constraints

The weak ideal boundary constraint gives the same result as if the weak constraints are turned off, except that using weak ideal constraints adds an additional dependent variable (a so-called Lagrange multiplier), which you can use for extremely accurate postprocessing.

If you select **Ideal** from the **Constraint type** list in the previous example, you get the same solution. If you then open the **Solver Manager** and click the **Solve For** tab, the list of variables to solve for contains an additional dependent variable. The name of this Lagrange multiplier variable is `lm1`. The Lagrange multiplier is available for postprocessing via boundary integration on boundaries and provides an accurate value for the integrated flux.

Weak Non-Ideal Constraints

The weak non-ideal boundary constraints gives a different solution. In this case the coupling is unidirectional, and it provides information transfer from the right boundary to the left boundary but not the other way around. The left boundary and the solution in the vicinity of the left boundary do not affect the solution at the right boundary. This type of unidirectional constraint can be explained intuitively as a controller that senses the average temperature at the right boundary and forces the temperature at the left boundary to take this value. The left boundary has, however, no way of telling the right boundary of its temperature.

In general, use non-ideal boundary constraints for automatic control design and similar applications, where values from one part of the solution are "forced upon" another part of the solution. On the other hand, use ideal boundary constraints for problems where the model is "left by itself," and the laws of nature balance the solution to an energy minimum (or, for problems with no well-defined energy, a saddle point).

Whenever using coupling variables in constraint expressions as in this example, make sure you do not use a different constraint at an adjacent boundary, because this would most likely create an ill-posed problem with no well-defined solution.

Theoretical Background

The mathematical weak form gives you direct access to the terms of the weak equation and provides maximum freedom in defining finite element problems. This section provides a theoretical background to the weak form in COMSOL Multiphysics.

Deriving the Weak Form

Before describing what the weak form is, this discussion first looks at the conversion of a simple problem in general form to the weak form.

Consider a stationary PDE problem for a single dependent variable, *u,* in two space dimensions:

$$\nabla \cdot \Gamma = F \qquad \text{in } \Omega$$

Now let v be an arbitrary function on Ω, and call it the *test function* (v should of course belong to a suitably chosen well-behaved class of functions, V). Multiplying the PDE with this function and integrating leads to

$$\int_{\Omega} v\nabla \cdot \Gamma dA = \int_{\Omega} vFdA$$

where dA is the area element. Now use Green's formula (Gauss' formula) to integrate by parts:

$$\int_{\partial\Omega} v\Gamma \cdot \mathbf{n}ds - \int_{\Omega} \nabla v \cdot \Gamma dA = \int_{\Omega} vFdA$$

where ds is the length element. Now use the Neumann boundary condition

$$-\mathbf{n} \cdot \Gamma = G + \frac{\partial R}{\partial u}\mu$$

to arrive at the following equation:

$$0 = \int_{\Omega} (\nabla v \cdot \Gamma + vF)dA + \int_{\partial\Omega} v\left(G + \frac{\partial R}{\partial u}\mu\right)ds$$

Together with the Dirichlet condition, this is a weak reformulation of the original PDE problem. Note that the requirement is that the above weak equation should hold for *all* test functions v. You can reverse the steps of the derivation to show that if the functions u and μ satisfy the weak formulation, then they also satisfy the original formulation. However, this holds true only if the solutions and coefficients are sufficiently smooth. For instance, in the case of discontinuities in material properties, you can have a solution of the weak formulation while the strong formulation has no meaning. The names "weak" and "strong" stem from this distinction: the weak formulation is a weaker condition on the solution than the strong formulation. A benefit of the weak formulation is that it requires less regularity of Γ. This is important in the finite element method.

For the most part, the weak reformulation of a system of equations in general form is

$$\begin{cases} 0 = \int\limits_{\Omega} (\nabla v_l \cdot \Gamma_l + v_l F_l) dA + \int\limits_{\partial\Omega} v_l \left(G_l + \dfrac{\partial R_m}{\partial u_l} \mu_m \right) ds \\ 0 = R_m \qquad \text{on } \partial\Omega \end{cases}$$

where you use the summation convention. If the boundary conditions are given on interior boundaries, $\partial\Omega$ can be enlarged to include these. You now have several test functions $v_1, v_2, \ldots, v_N$, one for each of the original PDEs. The weak equation is required to hold for all test functions (in the suitably chosen function space V). Of course, in three dimensions, for dA and ds you must substitute the volume and surface area elements, respectively. In one dimension, for dA substitute the length element, and interpret the boundary integral as a sum over the boundary points.

Weak Form Application Modes

In COMSOL Multiphysics you can add weak form PDEs on all domain levels. From the **PDE Modes** folder in the Model Navigator, select from:

- **Weak Form, Subdomain**
- **Weak Form, Boundary**
- **Weak Form, Edge** (3D only)
- **Weak Form, Point**

The Weak Form, Boundary application modes are of great use when you must define a PDE on a boundary and couple this equation to an equation in a subdomain. A typical example exists in adsorption processes when you are coupling a material balance of the adsorbed species on the boundary to the material balance of a species defined in the domain (see “Theoretical Background” on page 310).

In addition, you can add weak form contributions to other application modes on the **Weak** page in the **Subdomain Settings**, **Boundary Settings**, **Edge Settings**, and **Point Settings** dialog boxes.

For an example of how to use a weak form contribution to implement a point source, see “Implementing a Point Source” on page 279.

VARIATIONAL PRINCIPLES

This section provides some background on the weak formulation. You can eliminate the Lagrange multipliers from the weak equation in the previous section by introducing the following boundary conditions on the test functions:

$$0 = \frac{\partial R_m}{\partial u_l} v_l \qquad \text{on } \partial\Omega$$

This means that the functions $u_l + v_l$ satisfy the linearized version of the Dirichlet boundary conditions. Then the weak formulation becomes the following: find functions u_l such that

$$\begin{cases} 0 = \int\limits_{\Omega} (\nabla v_l \cdot \Gamma_l + v_l F_l) dA + \int\limits_{\partial\Omega} v_l G_l ds \\ 0 = R_m \qquad \text{on } \partial\Omega \end{cases}$$

holds for all test functions v_l that satisfy the above boundary condition. Such a formulation arises when you have a *variational principle*; for instance, to find the functions u_l that minimizes the energy of some physical system under the constraints $0 = R_m$. If the energy is given as an integral of some expression involving the u_l, then the stationarity condition on the solution is precisely the weak formulation above. Because variational principles are more fundamental than the corresponding PDE, the weak form is often more natural than the strong forms. The connection to variational principles also shows why the term involving the Lagrange multipliers takes the form it does (that is, how ideal constraints come about).

THE WEAK FORM PDE STRUCTURE

Now consider what a problem on the weak form can look like. It can be more general than the weak equations derived from the general form above because it also contains integrals over edges (in 3D) or points (in 2D and 3D). Similarly, you can also have constraints on subdomains, edges, and points. Further, expressions occurring in the integrands can be more or less arbitrary functions of the variables u_l, the test functions v_l, and their derivatives. One important restriction, though, is that the test functions enter linearly. A problem in weak form has the following structure (in 2D):

$$0 = \int_{\Omega} W^{(2)} dA + \int_{B} W^{(1)} ds + \sum_{P} W^{(0)} +$$

$$+ \int_{\Omega} v_l \frac{\partial R_m^{(2)}}{\partial u_l} \mu_m^{(2)} dA + \int_{B} v_l \frac{\partial R_m^{(1)}}{\partial u_l} \mu_m^{(1)} ds + \sum_{P} v_l \frac{\partial R_m^{(0)}}{\partial u_l} \mu_m^{(0)}$$

$$0 = R^{(2)} \quad \text{on } \Omega$$

$$0 = R^{(1)} \quad \text{on B}$$

$$0 = R^{(0)} \quad \text{on P}$$

The first equation is the weak equation, and the others are the constraints. Here Ω is the subdomain, B is the boundary (including ∂Ω and interior boundaries), and P represents the vertices (points) defined in the geometry. The integrands $W^{(i)}$ are scalar expressions involving the dependent variables $u_1, u_2,..., u_N$ as well as the test functions $v_1, v_2,..., v_N$ and their derivatives. You must enter the test functions and their derivatives linearly. For instance, you can have

$$W^{(2)} = v_1 \frac{\partial u_2}{\partial x}^2 + \frac{\partial v_2}{\partial x} \sin u_1 + \frac{\partial^2 v_1}{\partial y^2}$$

The quantities $R^{(i)}$ occurring in the constraints are vector-valued functions of the dependent variables. Sometimes they also depend on the derivatives of the dependent variables. In such cases there are additional terms accounting for this dependence in the integrals with Lagrange multipliers. Note that you now also can have Lagrange multipliers on subdomains ($\mu^{(2)}$) and on vertices ($\mu^{(0)}$).

In 3D, contributions to the weak equation and the constraints can also come from edges (curves).

CONVERSION TO WEAK FORM

It is possible to convert equations in coefficient form or general form to weak form by setting the equation system form to weak using the **Equation system form** list in the **Model Settings** dialog box, which opens from the **Physics** menu. To view the result of the conversion, go to the **Physics** menu, point to **Equation System** and then click on a domain type settings command. COMSOL Multiphysics also carries out this conversion conceptually in the finite element method. The following equations describe the obtained weak form problem as well as problems given directly in weak form. For simplicity this discussion considers a stationary problem in 2D.

$$0 = \int_\Omega W^{(2)} dA + \int_B W^{(1)} ds + \sum_P W^{(0)}$$

$$-\int_\Omega v \cdot h^{(2)^T} \mu^{(2)} dA - \int_B v \cdot h^{(1)^T} \mu^{(1)} ds - \sum_P v \cdot h^{(0)^T} \mu^{(0)}$$

$$0 = R^{(2)} \quad \text{on } \Omega$$

$$0 = R^{(1)} \quad \text{on B}$$

$$0 = R^{(0)} \quad \text{on P}$$

Here v is a vector whose components are the test functions v_l, while $\mu^{(d)}$ are column vectors of Lagrange multipliers. The matrices $h^{(d)}$ are

$$h^{(d)} = -\frac{\partial R^{(d)}}{\partial u}$$

for a problem given directly in weak form. For a problem originally given in coefficient form or general form, the vectors $R^{(2)}$ and $R^{(0)}$ as well as the Lagrange multipliers $\mu^{(2)}$ and $\mu^{(0)}$ are empty. In addition, $W^{(0)}=0$. For the general form you have

$$h^{(1)} = -\frac{\partial R}{\partial u}$$

For the coefficient form, you have $h^{(1)}=h$.

For a problem given directly in weak form, the constraints can also depend on variables other than the u_l (for instance the space derivatives of u_l). In such cases, the quantities $v \cdot h^{(d)^T}$ are augmented with additional terms that account for this dependence.

Conversion from general form to weak form is performed according to

$$W_l^{(n)} = W_l^{(n)} + \Gamma_{lj}\frac{\partial v_l}{\partial x_j} + F_l v_l$$

$$W_l^{(nt)} = W_l^{(nt)} + d_{alk}\frac{\partial u_k}{\partial t} v_l$$

$$W_l^{(n-1)} = W_l^{(n-1)} + G_l v_l$$

$$R_m^{(n)} = R_m$$

where there is an implicit summation over the k and i indices in each product. n is the space dimension. If you are working with COMSOL Script or MATLAB, see the entry `flform` in the *COMSOL Multiphysics Reference Guide* for more information about the conversion to weak form.

Entering a PDE in Coefficient Form Using the Weak Form

Derive the weak equation for a coefficient form problem:

$$0 = \int_{\Omega} (\nabla v \cdot (-c\nabla u - \alpha u + \gamma) + v(f - \beta \cdot \nabla u - au))dA + \int_{\partial\Omega} v(-qu + g - h^T \mu)ds$$

Suppose c is a scalar. Using the weak form meta variables for a 2D problem, follow these steps:

1 In the **weak** edit field of Ω in the **Subdomain Settings** dialog box, type

```
ux_test*(-c*ux-alx*u+gax)+uy_test*(-c*uy-aly*u+gay)+...
    u_test*(f-bex*ux-bey*uy-a*u)
```

2 In the **weak** field of ∂Ω in the **Boundary Settings** dialog box, type

```
u_test*(-q*u+g)
```

The variables `alx`, `aly`, `bex`, `bey`, `gax`, and `gay` are the components of α, β, and γ, respectively. You must replace the coefficients in this weak form expression (for example, `c`, `alx` and `q`) with appropriate expressions.

COMSOL Multiphysics automatically adds the Lagrange multiplier μ, while the constraint matrix h is specified indirectly by the constraints.

12

Multiphysics Modeling

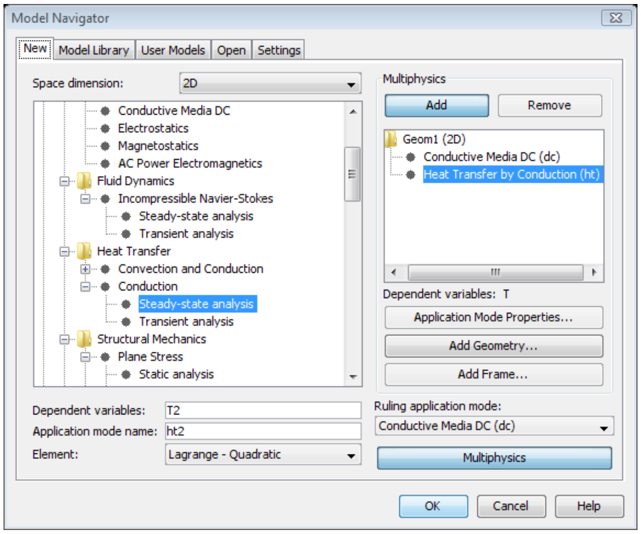
This chapter covers the use of COMSOL Multiphysics for multiphysics modeling and coupled-field analyses. It first describes the various ways of building multiphysics models. Then a step-by-step example shows how to model Joule heating (also called resistive heating) taking into account thermal-electric couplings.

Creating Multiphysics Models

Multiphysics Modeling Approaches

The ability to create multiphysics models—those with more than one type of physics or equation such as coupled-field problems—is one of COMSOL Multiphysics' most powerful capabilities. In such a model, the software can solve all the equations, taken from various areas of physics, as one fully coupled system.

Within COMSOL Multiphysics you can choose from several ways to approach multiphysics modeling and coupled-field analysis. They all use the **Model Navigator** to specify the application modes and their properties.



Space dimension: 2D

Dependent variables: T

Dependent variables: T2

Application mode name: ht2

Element: Lagrange - Quadratic

Ruling application mode: Conductive Media DC (dc)

Figure 12-1: The Model Navigator for a multiphysics model combining the Conductive Media DC and Heat Transfer by Conduction application modes.

Using Predefined Multiphysics Couplings

COMSOL Multiphysics and the add-on modules provide a number of predefined multiphysics couplings. They appear in the Model Navigator's list of application modes, in folders with the names in the Category/Folder column in Table 12-1. For each predefined multiphysics coupling, there is an entry in the Model Navigator in all products that support it. The predefined multiphysics couplings consist of two or more application modes that COMSOL Multiphysics adds to a model. In addition, default settings provide the typical coupled fields for the multiphysics application. The available predefined multiphysics couplings vary depending on the installed modules (see the following table for details).

TABLE 12-1: PREDEFINED MULTIPHYSICS COUPLINGS

CATEGORY/ FOLDER	NAME	SPACE DIMENSION	REQUIRED MODULES	OPTIONAL MODULES
Electro-Thermal Interaction	Joule Heating	2D, 2D Axi, 3D	None	AC/DC, Heat Transfer, MEMS
Electro-Thermal Interaction	Perpendicular Induction Heating	2D	AC/DC	Heat Transfer
Electro-Thermal Interaction	In-Plane Induction Heating	2D	AC/DC	Heat Transfer
Electro-Thermal Interaction	In-Plane Microwave Heating	2D	RF	Heat Transfer
Electro-Thermal Interaction	Meridional Induction Heating	2D Axi	AC/DC	Heat Transfer
Electro-Thermal Interaction	Azimuthal Induction Heating	2D Axi	AC/DC	Heat Transfer
Electro-Thermal Interaction	Meridional Microwave Heating	2D Axi	RF	Heat Transfer
Electro-Thermal Interaction	Induction Heating	3D	AC/DC	Heat Transfer
Electro-Thermal Interaction	Microwave Heating	3D	RF	Heat Transfer
Fluid-Chemical Reactions	Reacting Flow	2D, 2D Axi, 3D	Chemical Engineering	
Fluid-Structure Interaction	Plane Stress with Fluid Interaction	2D	Structural Mechanics or MEMS	Chemical Engineering
Fluid-Structure Interaction	Plane Strain with Fluid Interaction	2D	Structural Mechanics or MEMS	Chemical Engineering
Fluid-Structure Interaction	Axisymmetric Stress-Strain with Fluid Interaction	2D Axi	Structural Mechanics or MEMS	Chemical Engineering

TABLE 12-1: PREDEFINED MULTIPHYSICS COUPLINGS

CATEGORY/ FOLDER	NAME	SPACE DIMENSION	REQUIRED MODULES	OPTIONAL MODULES
Fluid-Structure Interaction	Solid, Stress-Strain with Fluid Interaction	3D	Structural Mechanics or MEMS	Chemical Engineering
Fluid-Thermal Interaction	Fluid-Thermal Incompressible Flow	2D, 2D Axi, 3D	None	Chemical Engineering, Heat Transfer
Fluid-Thermal Interaction	Non-Isothermal Fluid-Thermal Interaction	2D, 2D Axi, 3D	Chemical Engineering or Heat Transfer	
Fluid-Thermal Interaction	Turbulent Fluid-Thermal Interaction	2D, 2D Axi, 3D	Chemical Engineering or Heat Transfer	
Thermal-Structural Interaction	Plane Stress with Thermal Expansion	2D	Structural Mechanics	Heat Transfer
Thermal-Structural Interaction	Plane Strain with Thermal Expansion	2D	Structural Mechanics	Heat Transfer
Thermal-Structural Interaction	Axisymmetric Stress-Strain with Thermal Expansion	2D Axi	Structural Mechanics	Heat Transfer
Thermal-Structural Interaction	Solid, Stress-Strain with Thermal Expansion	3D	Structural Mechanics	Heat Transfer
Thermal-Structural Interaction	Shell with Thermal Expansion	3D	Structural Mechanics, Heat Transfer	
Structural Mechanics	Plane Strain with Film Damping	2D	MEMS	
Structural Mechanics	Solid, Stress-Strain with Film Damping	3D	MEMS	
Microfluidics	Flow with Species Transport	2D, 2D Axi, 3D	MEMS	

TABLE 12-1: PREDEFINED MULTIPHYSICS COUPLINGS

CATEGORY/ FOLDER	NAME	SPACE DIMENSION	REQUIRED MODULES	OPTIONAL MODULES
Microfluidics	Electroosmotic Flow	2D, 2D Axi, 3D	MEMS	
Rotating Machinery	Rotating Perpendicular Currents	2D	AC/DC	
Rotating Machinery	Rotating Navier-Stokes	2D	Chemical Engineering	

The predefined multiphysics coupling makes use of the application mode in the optional modules if your license includes them; otherwise the coupling uses the corresponding but more limited application mode in COMSOL Multiphysics. The couplings are identical in both cases, but the application mode from the module typically offers additional functionality in other areas. These predefined multiphysics couplings are quick entry points for common multiphysics applications. It is possible to create the same couplings using any of the other methods for multiphysics modeling, and you can continue to add, modify, and delete application modes in a model that you start using one of the predefined multiphysics couplings. If you want to add additional application modes directly in the **Model Navigator**, click the **Multiphysics** button and then the **Add** button. You can then see the application modes from the predefined multiphysics coupling in the **Multiphysics** area and select new application modes that you add to the model using the **Add** button.

In the **Model Navigator**, the **Dependent variables** list contains the dependent variables, and the **Application mode name** edit field contains the default application mode names for the application modes that the predefined multiphysics coupling includes. The **Element** list in the **Model Navigator** provides a selection of suitable element types for all the dependent variables. You can also select element types individually using the **Element** tab in the **Subdomain Settings** dialog box (and, in some cases, the **Boundary Settings** dialog box).

ELECTRO-THERMAL INTERACTION—JOULE HEATING

The Joule Heating predefined multiphysics coupling combines a Conductive Media DC application mode from COMSOL Multiphysics or the AC/DC Module with the Heat Transfer by Conduction application mode from COMSOL Multiphysics or the

General Heat Transfer application mode from the Heat Transfer Module. The interaction is fully coupled in both directions:

- In the heat transfer application mode, resistive heating appears as a heat source (the resistive heating variable `Q_es` (or `Q_emes`) appears in the **`Q`** edit field).
- In the Conductive Media DC application mode, the definition of the conductivity σ uses the thermal heating model $1/(\rho_0(1+\alpha(T-T_0)))$, where T is the dependent variable for temperature from the heat transfer application mode. The values for ρ_0 (resistivity at reference temperature), α (temperature coefficient), and T_0 (reference temperature) are default values that you can change to match your modeling situation.

You can select a steady-state or a transient analysis.

For an example of a model using the Joule Heating predefined multiphysics coupling, see “Example: Resistive Heating” on page 334.

ELECTRO-THERMAL INTERACTION—INDUCTION HEATING

The Induction Heating predefined multiphysics coupling combines an application mode for induction currents from the AC/DC Module with the Heat Transfer by Conduction application mode from COMSOL Multiphysics or the General Heat Transfer application mode from the Heat Transfer Module. The predefined interaction adds the resistive heating from the induction currents as a heat source in the application mode for the heat transfer.

For more information, see the *AC/DC Module User’s Guide*.

ELECTRO-THERMAL INTERACTION—MICROWAVE HEATING

The Microwave Heating predefined multiphysics coupling combines an application mode for electromagnetic waves from the RF Module with the Heat Transfer by Conduction application mode from COMSOL Multiphysics or the General Heat Transfer application mode from the Heat Transfer Module. The predefined interaction adds the resistive heating from the electromagnetic waves as a heat source in the application mode for the heat transfer.

For more information, see the *RF Module User’s Guide*.

FLUID-CHEMICAL REACTIONS INTERACTION—REACTING FLOW

The Reacting Flow predefined multiphysics coupling requires the Chemical Engineering Module. Reacting Flow combines a Convection and Diffusion mass balance application mode with an Incompressible Navier-Stokes application mode, both from the Chemical Engineering Module. The interaction is a one-way coupling using the velocity described by the Navier-Stokes equations as part of the convective mass transfer. The components of the velocity vector **u** from the Incompressible Navier-Stokes application mode appear in the **u**, **v**, and **w** edit fields in the **Subdomain Settings** dialog box for the Convection and Diffusion application mode.

By default, COMSOL Multiphysics solves for the velocity field, pressure, and species concentrations simultaneously. For large problems, you can take advantage of the one-way coupling and solve the problem sequentially (unless there are fluid-flow properties that depend on the species concentrations): First solve for velocities and pressures, and then solve the mass balance equation using the computed velocities from the Navier-Stokes equation. See "Solving for a Subset of the Dependent Variables" on page 329 on how to select which variables to solve for.

FLUID-STRUCTURE INTERACTION (FSI)

The Fluid-Structure Interaction (FSI) predefined multiphysics coupling is available in the MEMS Module and the Structural Mechanics Module. FSI combines fluid flow with structural mechanics, using a Moving Mesh (ALE) application mode to capture the fluid movement. Predefined groups of boundary settings help to define the fluid loads on the solid domain and the structural velocities affect on the fluid. For more information and example models, see the Structural Mechanics Module and MEMS Module documentation.

FLUID-THERMAL INTERACTION—FLUID-THERMAL INCOMPR. FLOW

The Fluid-Thermal Incompressible Flow predefined multiphysics coupling combines an Incompressible Navier-Stokes application mode from COMSOL Multiphysics with the Heat Transfer by Convection and Conduction application mode from COMSOL Multiphysics. The interaction is a one-way coupling using the velocity described by the Navier-Stokes equations as part of the convective heat transfer. The components of the velocity vector **u** from the Incompressible Navier-Stokes application mode appear in the **u**, **v**, and **w** edit fields in the **Subdomain Settings** dialog box for the heat transfer application mode.

By default, COMSOL Multiphysics solves for the velocity field, pressure, and species concentrations simultaneously. For large problems, you can take advantage of the one-way coupling and solve the problem sequentially (unless there are fluid-flow properties that depend on the temperature): first solve for velocities and pressures and then solve the heat transfer equation using the computed velocities from the Navier-Stokes equation. See "Solving for a Subset of the Dependent Variables" on page 329 on how to select which variables to solve for.

The Chemical Engineering Module and the Heat Transfer Module also include the Non-Isothermal Fluid-Thermal Interaction and Turbulent Fluid-Thermal Interaction predefined multiphysics couplings, which are similar but model non-isothermal fluids and turbulent fluid flow instead of incompressible, laminar fluid flow. For more information about these predefined couplings, see the Chemical Engineering Module and Heat Transfer Module documentation.

THERMAL-STRUCTURAL INTERACTION—THERMAL EXPANSION

The predefined multiphysics couplings for structural mechanics with thermal expansion requires the Structural Mechanics Module. They combine the Heat Transfer by Conduction application mode from COMSOL Multiphysics or the General Heat Transfer application mode from the Heat Transfer Module with the following application modes from the Structural Mechanics Module:

- Plane Stress in 2D
- Plane Strain in 2D
- Axial Symmetry, Stress-Strain in 2D axisymmetry
- Solid, Stress-Strain in 3D
- Shell in 3D

The interaction is a one-way coupling using the temperature described by a heat transfer application mode to define the strain temperature. On the **Load** page in the **Subdomain Settings** dialog box in the structural mechanics application mode, the **Include thermal expansion** check box is selected, and the dependent variable for temperature from the heat transfer application mode appears in the **Temp** edit field for the strain temperature. The **Tempref** edit field contains a default value (0) for the strain reference temperature. If necessary, adjust the strain reference temperature to a suitable value for your model.

By default, COMSOL Multiphysics solves for the temperature and displacements simultaneously. For large problems, you can take advantage of the one-way coupling and solve the problem sequentially (unless there are thermal properties that depend on the displacements): first solve for temperature and then perform the stress-strain analysis using the computed temperature field from the heat transfer equation. See “Solving for a Subset of the Dependent Variables” on page 329 on how to select which variables to solve for.

STRUCTURAL MECHANICS INCLUDING FILM DAMPING

The MEMS Module includes the Plane Strain with Film Damping and Solid, Stress-Strain with Film Damping predefined multiphysics couplings. The couple the structural mechanics application mode with the Film Damping application mode, for transient and frequency-response analysis of microsystems that include film damping. The Film Damping application mode models film damping on boundaries, and the predefined multiphysics coupling sets the boundary deformation to the displacement variables from the structural mechanics application mode.

For more information, see the MEMS Module documentation.

MICROFLUIDICS—FLOW WITH SPECIES TRANSPORT

This suite of predefined multiphysics couplings combines the application modes for microfluidics (General Laminar Flow, Incompressible Navier-Stokes, Non-Isothermal Flow, Stokes Flow, and Non-Isothermal Stokes Flow) with the Convection and Diffusion application mode. The coupling is similar to the one in the Reacting Flow predefined multiphysics coupling (see “Fluid-Chemical Reactions Interaction—Reacting Flow” on page 322), combining fluid flow with species (mass) transport.

For more information, see the MEMS Module documentation.

MICROFLUIDICS—ELECTROOSMOTIC FLOW

This suite of predefined multiphysics couplings for electroosmotic flow modeling combines the application modes for microfluidics (General Laminar Flow, Incompressible Navier-Stokes, Non-Isothermal Flow, Stokes Flow, and Non-Isothermal Stokes Flow) with the Conductive Media DC application mode.

Using the Electroosmotic Flow predefined multiphysics coupling, both application modes include boundary groups for defining boundaries as electrodes, ground, electroosmosis, inlets, and outlets.

For more information, see the MEMS Module documentation.

ROTATING MACHINERY—ROTATING PERPENDICULAR CURRENTS

For modeling of rotating machinery such as motors and generators, the Rotating Perpendicular Currents predefined multiphysics coupling combines the Perpendicular Induction Currents application mode with a Moving Mesh (ALE) application mode. The Moving Mesh (ALE) application mode, which captures the rotation, contains predefined groups with settings for fixed parts and parts that rotate clockwise or counterclockwise.

For more information, see the AC/DC Module documentation.

ROTATING MACHINERY—ROTATING NAVIER-STOKES

For modeling of rotating machinery with fluids (such as stirrers), the Rotating Navier-Stokes predefined multiphysics coupling combines the Incompressible Navier-Stokes application mode with a Moving Mesh (ALE) application mode. The Moving Mesh (ALE) application mode, which captures the rotation, contains predefined groups with settings for fixed parts and parts that rotate clockwise or counterclockwise.

For more information, see the Chemical Engineering Module documentation.

Adding Physics Sequentially

With this approach you can verify that each type of physics or PDE gives the expected results before adding more complexity to the model by adding another physics or coupling fields.

Use these steps to add one application mode at the time:

1 Select an application mode.
2 Draw the geometry.
3 Define the physics settings.
4 Solve and visualize the results.

   Then add another application mode:

5 Go to the **Multiphysics** menu and choose **Model Navigator**.
6 Select a new application mode.
7 In the **Multiphysics** area, click the **Add** button.
8 Click **OK**.
9 Add and modify settings in the **Physics** menu, including couplings between the physics.

Building a Coupled Multiphysics Model Directly

Another approach is to include multiple application modes when you start creating the model in the **Model Navigator**:

1 Select an application mode for the model.

2. Click the **Multiphysics** button.
3. In the **Multiphysics** area, click the **Add** button.
4. Continue selecting application modes and adding them to the model by clicking **Add**.
5. Click **OK**.

Removing Application Modes

To remove an application mode from a model:

1. Choose **Model Navigator** from the **Multiphysics** menu.
2. Select the application mode from the list under **Multiphysics**. You can also select a geometry in the list to remove the entire geometry and all application modes that it contains.
3. Click **Remove**.
4. Click **OK**.

Multiphysics Model Properties

When you add or remove application modes, COMSOL Multiphysics updates some model properties to:

- Include all dependent variables in the added application modes in the list of variables to solve for.
- Update all application-dependent dialog boxes.

ACTIVE APPLICATION MODE

The application mode selected under **Multiphysics** becomes the active application mode when you begin modeling.

To shift between the different application modes during modeling, select one from the **Multiphysics** menu to make it active.

RULING APPLICATION MODE

The *ruling application mode* determines the global settings for the multiphysics model from its application mode properties. For example, if the analysis type is different for two application modes in the same model, the software takes the analysis type from the ruling application mode. You select the ruling application mode for a multiphysics model in the Model Navigator.

Managing the Solver and Solution Components

The Solver Manager dialog box provides several options for managing the solution process for multiphysics analyses.

RESTARTING THE SOLVER AND USING INITIAL VALUES

When adding equations and physics to an existing model, you can restart the model using the current solution as an initial value. Doing so can accelerate the convergence for a coupled multiphysics model. To restart the solver, click the **Restart** button on the Main toolbar.

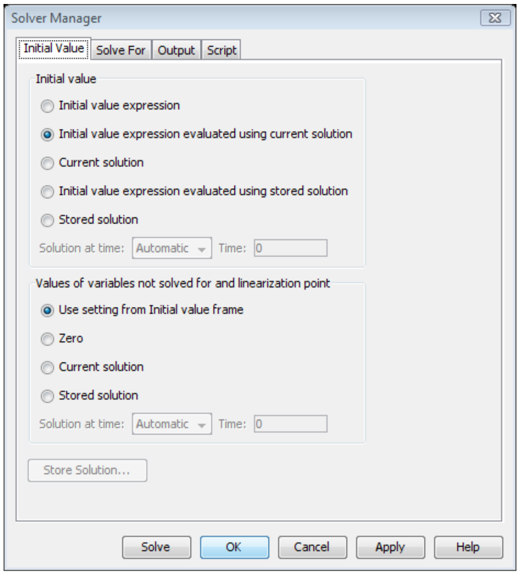
To specify initial values:

1 Go to the **Solve** menu and choose **Solver Manager**.

2 In the **Solver Manager** dialog box click the **Initial Value** tab.

3 Click the **Initial value expression using current solution** button (the default setting) to evaluate the expressions for the initial values as specified for each application mode using the current solution.

4 Click **OK**.



Solver Manager

Initial Value | Solve For | Output | Script

Initial value
- ( ) Initial value expression
- (●) Initial value expression evaluated using current solution
- ( ) Current solution
- ( ) Initial value expression evaluated using stored solution
- ( ) Stored solution

Solution at time: Automatic Time: 0

Values of variables not solved for and linearization point
- (●) Use setting from Initial value frame
- ( ) Zero
- ( ) Current solution
- ( ) Stored solution

Solution at time: Automatic Time: 0

Store Solution...

Solve | OK | Cancel | Apply | Help

Figure 12-2: The Solver Manager settings for evaluation the initial value expressions using the current solution.

SOLVING FOR A SUBSET OF THE DEPENDENT VARIABLES

The solver normally solves for all dependent variables. It is also possible to have it solve for a subset of the dependent variables. The initial conditions then define the value of the other variables. To specify which variables to solve for:

1 Go to the **Solve** menu and choose the **Solver Manager**.
2 In the **Solver Manager** dialog box click the **Solve For** tab.
3 Within the **Solve for variables** list the software selects all application modes by default. Clear the selections for those application modes whose dependent variables you do not want to solve for.
4 To select individual dependent variables for an application mode, click the plus sign to open the application mode folder and select the dependent variables to include.
5 Click **OK**.

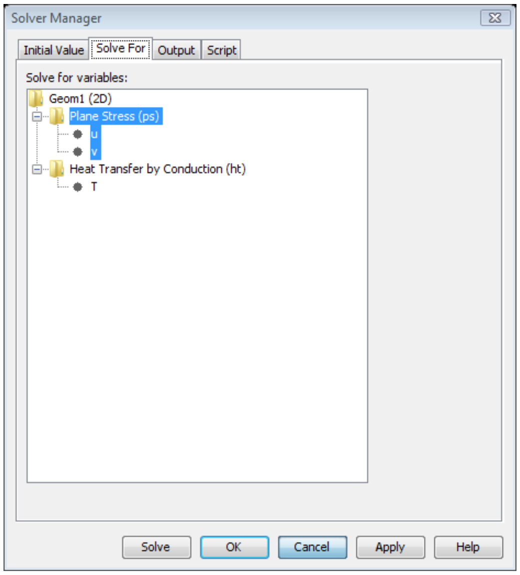


Figure 12-3: The Solve For page of the Solver Manager set up to solve for the displacements in the Plane Stress application model only.

Use this subset approach to reduce the computational load when there is only a one-way coupling in the multiphysics model:

1 Solve for the variables that are not affected by the coupling.

2 Solve for the remaining variables, using the solution from the initial analysis by restarting the solver using the previous solution data.

For a fully coupled multiphysics model with bidirectional couplings you can also use this approach to reach convergence in an iterative manner. Solve for one type of physics at a time and then use that solution as the initial value when solving for the other type of physics. This method provides a "weakly coupled" way of solving a coupled multiphysics model. Using the standard COMSOL Multiphysics approach to solving fully coupled equations simultaneously normally provides better convergence.

USING VARIABLES DURING ANALYSIS ONLY

In a similar way you can choose to output the results for a subset of the dependent variables:

1 In the **Solver Manager** dialog box click the **Output** tab.
2 In the **Output for variables** list select from the application modes and their dependent variables in the same way as for the variables to solve for.
3 Click **OK**.

This method can be useful if you want to include variables for multiphysics couplings during the analysis but are not interested in using them for postprocessing.

If you include nonsolved variables for output they take on the specified initial value as their solution data in the output from the solver.

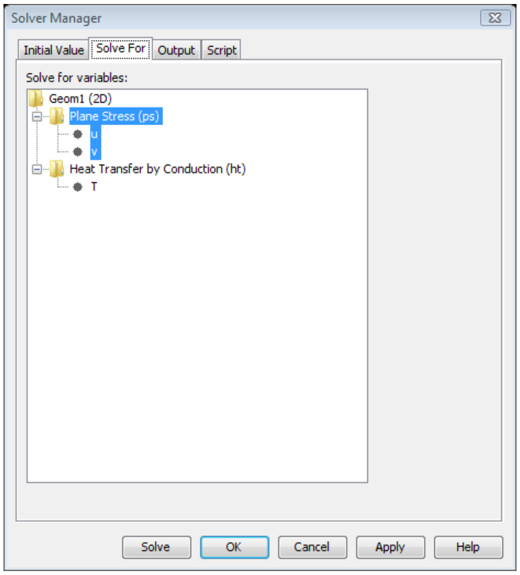


Figure 12-4: The Output page of the Solver Manager. By default, COMSOL Multiphysics includes all dependent variables from all application modes in the results data.

Deactivating Physics or Equations

Sometimes subdomains have different types of physics associated with them, and an application mode might not be meaningful in a subdomain. For example, fluid flow can be present only in some subdomains of a model whereas heat transfer takes place in the entire geometry. By default all application modes are active in all subdomains. To deactivate an application mode, select the subdomains where the application mode's physics do not take place. Then clear the **Active in this subdomain** check box in the **Subdomain Settings** dialog box and click **Apply**. Some interior boundaries then become exterior boundaries, and some of the original boundaries are absent for certain application modes.

Using the Model Navigator for Multiphysics Models

To add and remove application modes and geometries for a multiphysics model, go to the **Multiphysics** menu and then open the **Model Navigator**. Use its **Multiphysics** area to make changes to the current model's multiphysics properties.

ADDING GEOMETRIES TO A MODEL

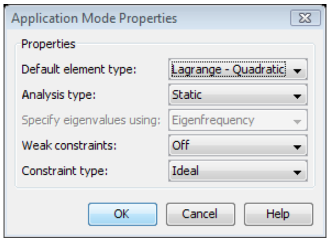
A model always has at least one geometry. To add another geometry to a model, click the **Add Geometry** button in the **Multiphysics** area of the **Model Navigator**. In such an *extended multiphysics* model the geometries can have different space dimensions. You can also choose the names of the independent variables (spatial coordinates).

CHANGING APPLICATION MODE PROPERTIES

Before adding an application mode, you can edit its name and dependent variables. COMSOL Multiphysics provides a unique name for each application mode in the model to identify the origin of the variables. You can change the name to any other name that is unique within the model.

You can also specify the element type, although the default is usually a good choice.

To set the analysis type and other application mode properties for the active application mode in a multiphysics model, click the **Application Mode Properties** button. These properties vary depending on the application mode.


Application Mode Properties

Properties

Default element type: Lagrange - Quadratic

Analysis type: Static

Specify eigenvalues using: Eigenfrequency

Weak constraints: Off

Constraint type: Ideal

Figure 12-5: The Application Mode Properties dialog box. The contents vary depending on the active application mode.

Specifying Settings for Multiphysics Models

In multiphysics models with multiple application modes, you can work with the settings for more than one application mode at a time. Selecting a settings command from the **Physics** menu opens the dialog box for the current application mode. All application modes in the model are listed in the **Multiphysics** menu. In that list, a bullet indicates the current application mode. To open a settings dialog box for another application mode, make it current by choosing it from the **Multiphysics** menu. You can now open the physics settings dialog boxes for this application mode. Any open physics settings dialog boxes for other application mode remain open and available for model specification.

THE EQUATION SYSTEM VIEW

The physics application modes and the application modes in the optional COMSOL modules provide built-in equations and the convenient specification of the physics that the application mode covers. Instead of entering PDE coefficients and general boundary coefficients, you select from typical material properties, sources, boundary conditions, and other physics settings. To find details on what these application modes provide, see the *COMSOL Multiphysics Modeling Guide* as well as the documentation that accompanies the optional modules.

COMSOL Multiphysics, however, always presents a model in terms of the underlying system of PDEs, the *equation system view*. In the **Physics** menu, point to **Equation System** and then open the dialog box for subdomain, boundary or point settings. You can also open these dialog boxes by Ctrl-clicking the corresponding selection mode button on the Main toolbar.

The equation system is the underlying representation of any model as a system of one or more PDEs in the coefficient or general forms. This representation is useful for several reasons:

- To let you introduce additional terms to a predefined physics mode
- To let you modify the equation, boundary conditions, and other settings from a predefined physics mode
- To add multiphysics couplings on the system level
- To check how COMSOL Multiphysics translates the equation and other settings in terms of physics

For more about the equation system views, see “Viewing and Modifying the Full Equation System” on page 213 and “Modifying Boundary Settings for the Equation System” on page 237 in the *COMSOL Multiphysics User’s Guide*.

Example: Resistive Heating

For an example of a multiphysics model with thermal-electric couplings, look at resistive heating in a copper plate.

Introduction

The material heats up when an electric current passes through it due to electric resistance. This is called resistive heating or Joule heating. There is also a coupling working in the opposite direction: the material's electric resistance varies with the temperature, increasing as the material heats up.

Model Definition

Imagine a copper plate measuring 1 m × 1 m that also contains a small hole. The plate's thickness has no effect on the model. Suppose that you subject the plate to an electric potential difference across two opposite sides (all other sides are insulated). The potential difference induces a current that heats the plate.

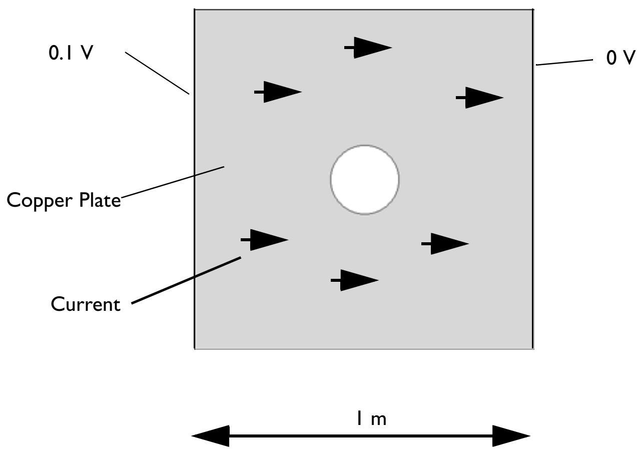


Figure 12-6: The model geometry and electric boundary conditions.

In the 2D model you view the plate from above.

Modeling in COMSOL Multiphysics

This example sets up the coupled thermal-electric analysis using the predefined multiphysics coupling for Joule Heating, which combines a Conductive Media DC application mode with a Heat Transfer by Conduction application mode. The potential distribution in the conductive media occurs almost instantly, but because unsteady heat transfer is a transient phenomenon, the full multiphysics model uses a transient analysis.

ADDITIONAL MULTIPHYSICS COUPLINGS

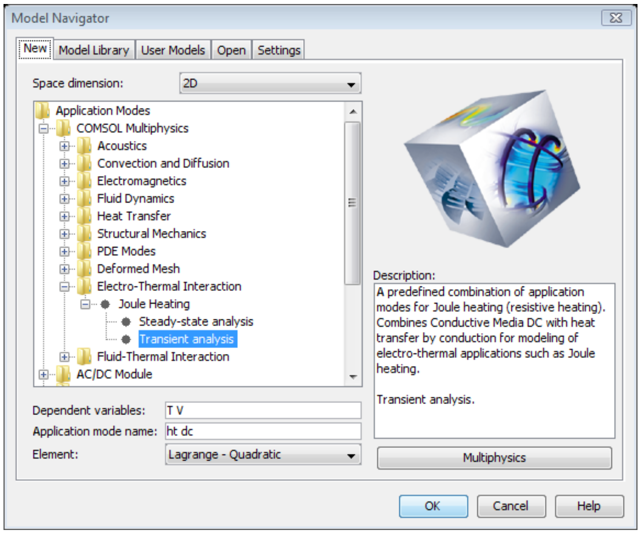
By adding a structural mechanics application mode to this model you could represent a thermomechanical coupling with a body force proportional to the temperature gradient. For more information on thermomechanical multiphysics, see the model "Simulation of a Microrobot" on page 389 in the *COMSOL Multiphysics Model Library* and the documentation for the Structural Mechanics Module.

Model Library path: `COMSOL_Multiphysics/Multiphysics/resistive_heating`

Modeling Using the Graphical User Interface

MODEL NAVIGATOR

1 On the **New** page, select **2D** from the **Space dimension** list.
2 In the list of application modes, open the **COMSOL Multiphysics>Electro-Thermal Interaction** folder and then the **Joule Heating** folder. Select **Transient analysis**.
3 Click **OK**.



Space dimension: 2D

- Application Modes
  - COMSOL Multiphysics
    - Acoustics
    - Convection and Diffusion
    - Electromagnetics
    - Fluid Dynamics
    - Heat Transfer
    - Structural Mechanics
    - PDE Modes
    - Deformed Mesh
    - Electro-Thermal Interaction
      - Joule Heating
        - Steady-state analysis
        - Transient analysis
    - Fluid-Thermal Interaction
  - AC/DC Module

Description:
A predefined combination of application modes for Joule heating (resistive heating). Combines Conductive Media DC with heat transfer by conduction for modeling of electro-thermal applications such as Joule heating.

Transient analysis.

Dependent variables: T V
Application mode name: ht dc
Element: Lagrange - Quadratic

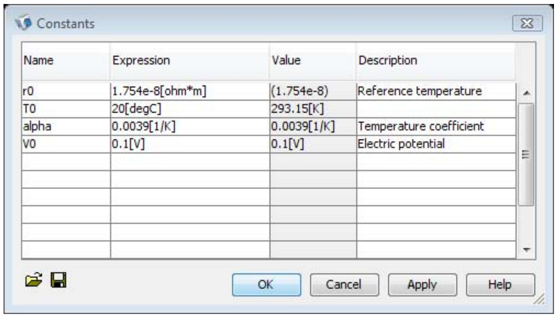
OPTIONS AND SETTINGS

1 From the **Options** menu choose **Constants**.

2 Enter the following constant names, expressions, and (optionally) descriptions:

NAME	EXPRESSION	DESCRIPTION
`r0`	`1.754e-8[ohm*m]`	`Resistivity at reference temperature`
`T0`	`20[degC]`	`Reference temperature`
`alpha`	`0.0039[1/K]`	`Temperature coefficient`
`V0`	`0.1[V]`	`Electric potential`

Enter the constant's name in the **Name** edit field, then place its value and unit in the **Expression** field, and finally add a description in the **Description** edit field:



Name	Expression	Value	Description
r0	1.754e-8[ohm*m]	(1.754e-8)	Reference temperature
T0	20[degC]	293.15[K]	
alpha	0.0039[1/K]	0.0039[1/K]	Temperature coefficient
V0	0.1[V]	0.1[V]	Electric potential

3 Click **OK**.

GEOMETRY MODELING

Define the model geometry:

1 Shift-click the **Rectangle/Square (Centered)** toolbar button.

2 In the **Square** dialog box, click **OK** to use the default values and create a unit square with corners at (0, 0) and (1, 1).

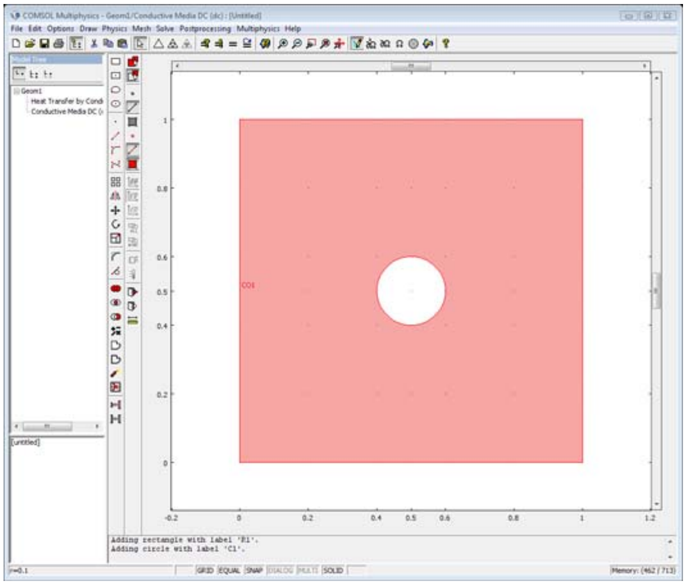
3 Click the **Zoom Extents** toolbar button (on the Main toolbar).

4 Shift-click the **Ellipse/Circle (Centered)** toolbar button.

5 In the **Circle** dialog box, type `0.1` in the **Radius** edit field and then type `0.5` in the **x** and **y** edit fields in the **Position** area. Click **OK**.

6 To create a hole, select both geometry objects by pressing Ctrl+A.

7 Click the **Difference** button on the Draw toolbar.


COMSOL Multiphysics - Geom1/Conductive Media DC (dc) : [Untitled]
File Edit Options Draw Physics Mesh Solve Postprocessing Multiphysics Help
Model Tree
Geom1
Heat Transfer by Cond
Conductive Media DC (
[untitled]
CO1
1
0.8
0.6
0.5
0.4
0.2
0
-0.2
0
0.2
0.4
0.5
0.6
0.8
1
1.2
r=0.1
GRID EQUAL SNAP DIALOG MULTI SOLID
Memory: (462 / 713)

```
Adding rectangle with label 'R1'.
Adding circle with label 'C1'.
```

PHYSICS SETTINGS

The electric boundary conditions appear in Figure 12-6. For the thermal boundary conditions, an air stream at 300 K (27 °C) cools the plate except on the thermally insulated upper and lower edges:

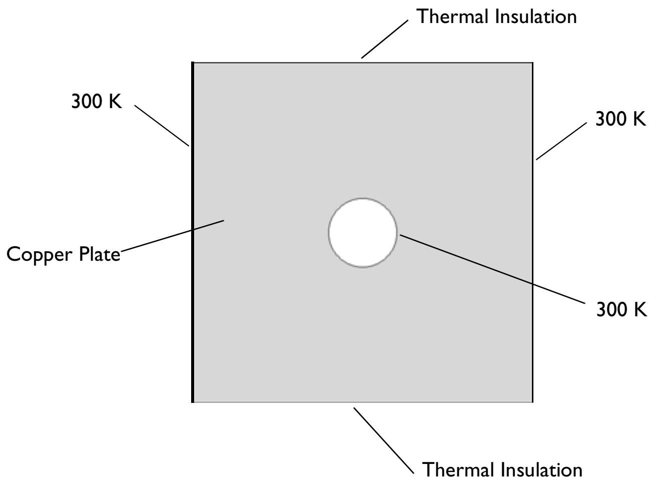


Figure 12-7: The model geometry and the thermal boundary conditions.

In Joule heating, the temperature increases due to the resistive heating from the electric current. The electric potential V is the solution variable in the Conductive Media DC application mode. The generated resistive heat Q is proportional to the square of the magnitude of the electric current density J. Current density, in turn, is proportional to the electric field, which equals the negative of the gradient of the potential V:

$$Q \propto |J|^2$$

The coefficient of proportionality is the electric resistivity $\rho = 1/\sigma$, which is also the reciprocal of the temperature-dependent electric conductivity $\sigma = \sigma(T)$. Combining these facts gives the fully coupled relation

$$Q = \frac{1}{\sigma}|J|^2 = \frac{1}{\sigma}|\sigma E|^2 = \sigma|\nabla V|^2$$

This resistive heating source term is directly available as the variable `Q_dc` (`Q_emdc` if you use the AC/DC Module) and is predefined as the source term in the heat transfer application mode when using the Joule Heating predefined multiphysics coupling.

Quantities other than σ also vary with temperature. For example, the thermal conductivity is temperature dependent, and a refined model would take this into account.

Over a range of temperatures the electric conductivity σ is a function of temperature T according to

$$\sigma = \frac{\sigma_0}{1 + \alpha(T - T_0)}$$

where σ_0 is the conductivity at the reference temperature T_0. α is the temperature coefficient of resistivity, which describes how the resistivity varies with temperature. A typical value for copper is 0.0039 per kelvin.

In the Conductive Media DC application mode you can specify the electric conductivity for Joule heating in terms of this equation. The predefined multiphysics coupling sets up this specification of sigma with the variable for temperature in the **T** edit field and default values suitable for copper. The only thing you have to add is the reference temperature, T_0.

Boundary Conditions—Heat Transfer

1 Make sure that **Heat Transfer by Conduction** or **General Heat Transfer** (if you use the Heat Transfer Module) is the selected application mode in the **Multiphysics** menu.

2 Open the **Boundary Settings** dialog box from the **Physics** menu.

The thermal boundary conditions are:

SETTINGS	BOUNDARIES 1, 4–8	BOUNDARIES 2, 3
Type	Temperature	Thermal insulation
T_0	300	

3 Select all boundaries.

4 Select **Temperature** from the **Boundary condition** list.

5 Type 300 in the **T$_0$** (temperature) edit field. This value corresponds to holding all boundaries at 300 K by positioning them in an air stream at room temperature.

6 Select Boundaries 2 and 3, the top and bottom boundaries, and then select Thermal insulation from the **Boundary condition** list.

7 Click **OK**.

Subdomain Settings—Heat Transfer

The material properties for heat transfer are:

PROPERTY	VALUE
ρ	8930
C_p	340
k (isotropic)	384
Q	Q_dc

1 From the **Physics** menu choose **Subdomain Settings**.

2 Select Subdomain 1 from the **Subdomain selection** list.

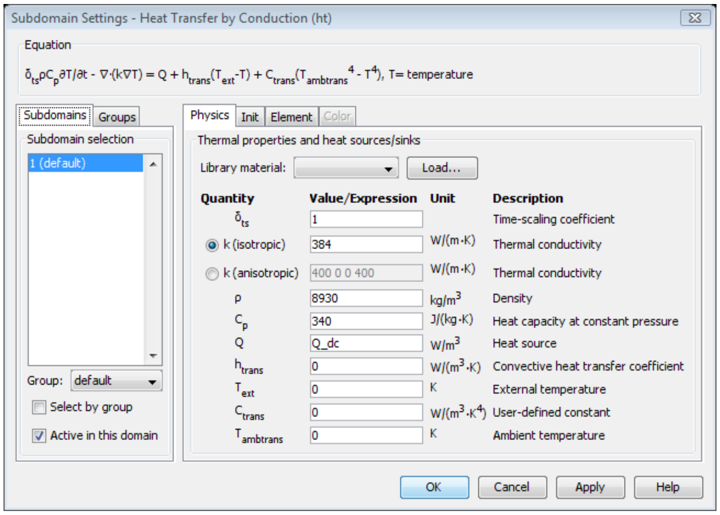
3 Type 8930 in the **Density** edit field (the density of copper in kg/m^3).

4 Type 340 in the **Heat capacity** edit field. The unit for heat capacity in this model is J/(kg·K), the SI unit.

5 Type 384 in the **k (isotropic)** edit field for the thermal conductivity.

This coefficient has the SI unit W/(m·K) and represents the material's ability to conduct heat per unit time.

6 The predefined setting in the **Heat source** edit field is Q_dc (or Q_emdc). This is a predefined variable for the resistive heating from the Conductive Media DC application mode and includes the temperature-dependent conductivity and the gradient of the electric potential.



Subdomain Settings - Heat Transfer by Conduction (ht)

Equation

$\delta_{ts}\rho C_p \partial T/\partial t - \nabla\cdot(k\nabla T) = Q + h_{trans}(T_{ext}-T) + C_{trans}(T_{ambtrans}{}^4 - T^4)$, T= temperature

Subdomain selection

1 (default)

Group: default

☐ Select by group

☑ Active in this domain

Thermal properties and heat sources/sinks

Library material:

Quantity	Value/Expression	Unit	Description
δ_{ts}	1		Time-scaling coefficient
● k (isotropic)	384	W/(m·K)	Thermal conductivity
○ k (anisotropic)	400 0 0 400	W/(m·K)	Thermal conductivity
ρ	8930	kg/m^3	Density
C_p	340	J/(kg·K)	Heat capacity at constant pressure
Q	Q_dc	W/m^3	Heat source
h_{trans}	0	W/(m^3·K)	Convective heat transfer coefficient
T_{ext}	0	K	External temperature
C_{trans}	0	W/(m^3·K^4)	User-defined constant
$T_{ambtrans}$	0	K	Ambient temperature

Initial Conditions—Heat Transfer

Also enter the initial value for the temperature, which is 300 K just as the boundary condition:

1 Click the **Init** tab in the **Subdomain Settings** dialog box.
2 Type `300` as the initial value in the edit field for **T(t$_0$)**.
3 Click **OK**.

Application Mode Selection

To define the settings for the copper plate's electric properties, switch to the Conductive Media DC application mode, selecting it from the **Multiphysics** menu or using the Model Tree.

Boundary Conditions—Conductive Media DC

1 Open the **Boundary Settings** dialog box by choosing **Boundary Settings** from the **Physics** menu. The boundary conditions for the Conductive Media DC application mode in this example are:

SETTINGS	BOUNDARY 1	BOUNDARY 4	BOUNDARIES 2, 3, 5–8
Type	Electric potential	Ground	Electric insulation
V	V0		

2 Press Ctrl+A to select all boundaries.
3 From the **Boundary condition** list select **Electric insulation**.

   Change the conditions on Boundaries 1 and 4:
4 From the **Boundary selection** list select Boundary 1, the left boundary.
5 From the **Boundary condition** list select **Electric potential** and type `V0` in the **V$_0$** edit field.
6 Select Boundary 4.
7 From the **Boundary condition** list select **Ground**.
8 Click **OK**.

Subdomain Settings—Conductive Media DC

1 From the **Physics** menu choose **Subdomain Settings**.
2 Select Subdomain 1 from the **Subdomain selection** list.
3 From the **Conductivity relation** list, select **Linear temperature relation**.
4 Make sure that the **T** edit field contains the field variable for temperature, `T`.
5 Type `r0` in the ρ_0 edit field for the resistivity at the reference temperature.

6 Type `alpha` in the α edit field for the temperature coefficient.

7 Type `T0` in the **T_0** edit field for the reference temperature.

8 Click **OK**.

Initial Conditions—Conductive Media DC

1 In the **Subdomain Settings** dialog box, click the **Init** tab.

2 Type `V0*(1-x[1/m])` in the edit field for **$V(t_0)$**. This expression means that the initial potential distribution varies linearly from V_0 (0.1 V) at the left boundary (x = 0) to `0` V at the right boundary (x = 1). This is an initial condition that matches the boundary conditions. The reason for the multiplying `x` with the unit `[1/m]` is to make it dimensionless.

3 Click **OK**.

MESH GENERATION

Click the **Initialize Mesh** button to create a mesh using the default settings.

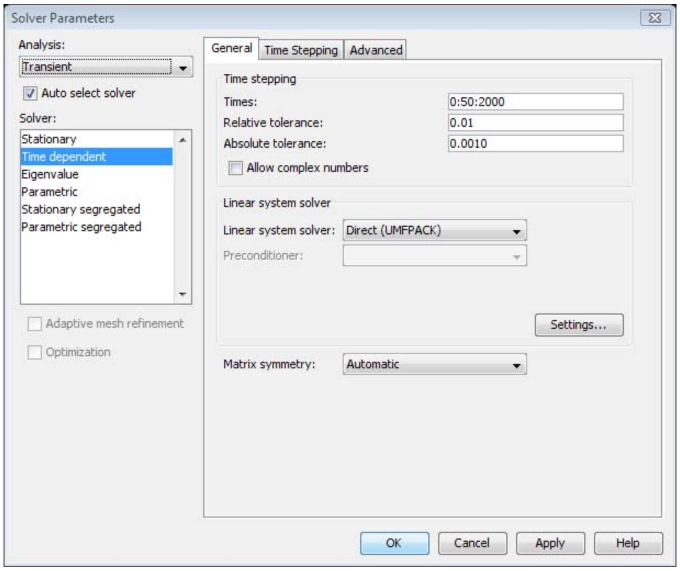
COMPUTING THE SOLUTION

The heat transfer in this model is a transient process, so the model uses a time-dependent solver for a transient analysis. First specify the simulation's time interval and the points in time to present solution data.

1 Open the **Solver Parameters** dialog box from the **Solve** menu.

2 Type `0:50:2000` in the **Times** edit field. Doing so produces a vector of 41 output times, linearly distributed every 50 seconds from 0 to `2000` seconds, for which the solution is available for postprocessing.

3 Click **OK**.


Solver Parameters

Analysis: Transient

☑ Auto select solver

Solver:

- Stationary
- Time dependent
- Eigenvalue
- Parametric
- Stationary segregated
- Parametric segregated

☐ Adaptive mesh refinement

☐ Optimization

General | Time Stepping | Advanced

Time stepping

Times:	0:50:2000
Relative tolerance:	0.01
Absolute tolerance:	0.0010

☐ Allow complex numbers

Linear system solver

Linear system solver: Direct (UMFPACK)

Preconditioner:

Settings...

Matrix symmetry: Automatic

OK Cancel Apply Help

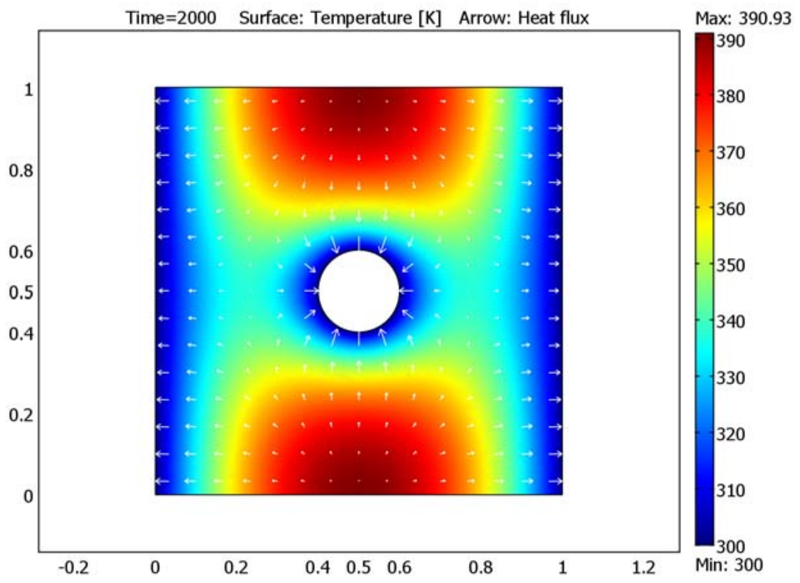
COMPUTING THE SOLUTION

You can now run the transient analysis. To do so, click the **Solve** button on the Main toolbar.

POSTPROCESSING AND VISUALIZATION

The default plot from this analysis shows the temperature or the potential distribution.

Visualizing Heat Flux Using Arrow Plots



A quantity that is interesting to animate is the heat flux:

1 Open the **Plot Parameters** dialog box from the **Postprocessing** menu.
2 Click the **Arrow** tab.
3 Select the **Arrow plot** check box.
4 Select **Heat Transfer by Conduction (ht)>Heat flux** from the **Predefined quantities** list.
5 Click the **Color** button and select a color for the arrows, for example, white.
6 Click **OK**.

Checking Heat Transfer Dynamics

Use a plot of temperature over time to determine if the time span for the simulation is sufficient to reach steady state:

1 From the **Postprocessing** menu choose **Cross-Section Plot Parameters**.
2 Make sure to select all time steps in the **Solutions to use** list on the **General** page.
3 Click the **Point plot** button.
4 Click the **Point** tab.
5 Select **Heat Transfer by Conduction (ht)>Temperature** in the **Predefined quantities** list.

6 In the **Coordinates** area, type `0.5` in the **x** edit field and `0.75` in the **y** edit field.

7 Click **OK**.

The plot of temperature as a function of time shows that it has reached a steady state. The temperature hardly increases at all if you extend the simulation over a longer time span.

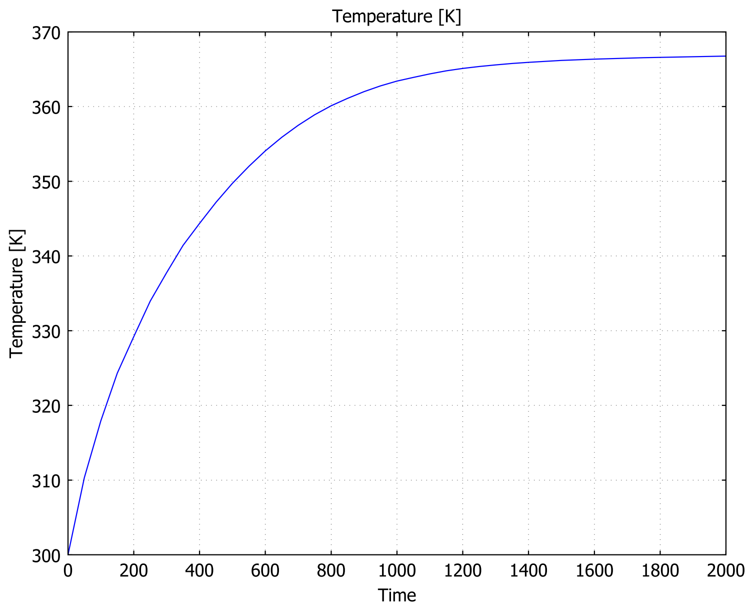


Figure 12-8: Heat transfer dynamics visualized as temperature versus time.

COMPUTING THE SOLUTION USING A SEGREGATED STATIONARY SOLVER

You can also solve this model for the steady-state condition. There is no problem solving this coupled multiphysics model using the standard stationary solver because the model is small. For large models, a segregated solver approach can be beneficial. This example shows how to solve a multiphysics model using the segregated stationary solver. You continue with the Resistive Heating model, adding the following steps:

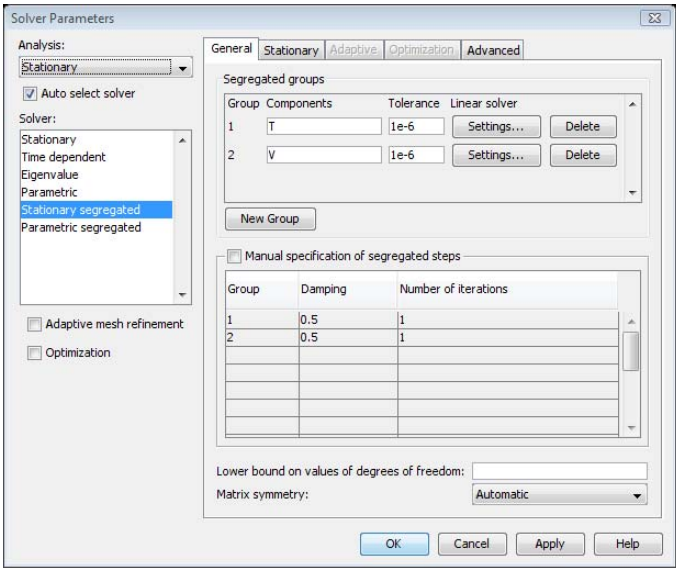
1 From the **Solve** menu, choose **Solver Parameters**.

2 Select **Stationary** from the **Analysis** list.

3 Select **Stationary segregated** from the **Solver** list. COMSOL Multiphysics has two predefined groups, one for each application mode (the temperature T and the electric potential V). All you need to do is to specify the tolerances so that the

segregated solver scheme converges to the same solution as the standard stationary solver. Use the default linear solver settings (the direct solver UMFPACK).

4 Type `1e-6` in the **Tolerance** edit field for both groups (Group 1 for T and Group 2 for V). The following figure shows the settings for the stationary segregated solver.



Group	Components	Tolerance	Linear solver
1	T	1e-6	Settings...
2	V	1e-6	Settings...

Group	Damping	Number of iterations
1	0.5	1
2	0.5	1

Lower bound on values of degrees of freedom:

Matrix symmetry: Automatic

5 Click **OK**.

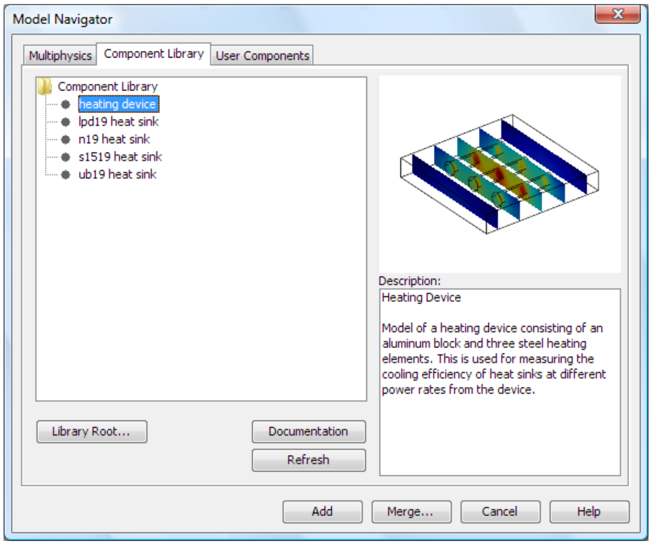
6 Click the **Solve** button on the Main toolbar.

Working with Components

A multiphysics model can consist of one or several components. Each component can have material data and physics. To build a model consisting of several components you can build one component at a time and save it in a Model MPH-file. When you have all the components you can start to put them together by merging them with or adding them to the current model.

The Component Library

To store Model MPH-files for use as components, you can use the Component Library. To open the Component Library, choose **File>Open Component Library**, or open the **Model Navigator** from the **Multiphysics** menu. Click the **Component Library** tab to see the components shipped with COMSOL Multiphysics.



Description:

Heating Device

Model of a heating device consisting of an aluminum block and three steel heating elements. This is used for measuring the cooling efficiency of heat sinks at different power rates from the device.

Figure 12-9: The Component Library in the Model Navigator.

The **User Components** page is available for selecting those components you have created. Click the **Library Root** button to specify the root directory for your own component libraries.

Adding Components

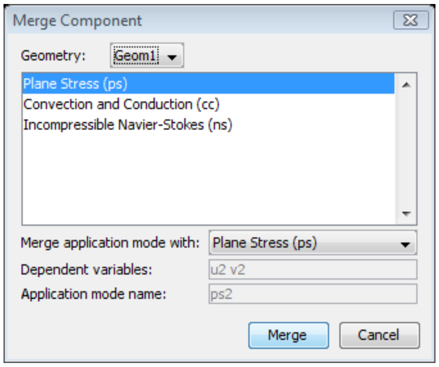
To add a component to a model on which you are working, select **Add Component** from the **File** menu; alternately, select the component you want to add in the Component Library and click the **Add** button. You can also add a component from the Model Tree when you choose the Detail and Inspect views. Right-click the top node with the model's name, and then select **Add Component**.

COMSOL Multiphysics creates a new geometry page and adds the component to this geometry. If the component has several geometries, the software adds all the geometries to the model. Adding a component includes the component's physics and mesh but not the solution.

Merging Components

To merge a component to the model on which you are working, either select **Merge Component** from the **File** menu or select the component you want to merge in the **Component Library** and click the **Merge** button. You can also add a component from the Model Tree. Right-click the geometry node where you want to merge a component with the current mode, and then select **Merge Component**.

When merging a component the **Merge Component** dialog box appears.


Merge Component

Geometry: Geom1

Plane Stress (ps)
Convection and Conduction (cc)
Incompressible Navier-Stokes (ns)

Merge application mode with: Plane Stress (ps)
Dependent variables: u2 v2
Application mode name: ps2

In the **Geometry** list you can find the geometries in the component. Select the geometry you want to merge into the model. Clicking **Merge** merges the geometry objects in the selected geometry into the current geometry in your model. The resulting geometry becomes an assembly.

When you select a geometry in the dialog box, the application modes in this geometry appear in a list. If application modes of the same type as the selected one already exist in the model, you can choose to merge the component's application mode with one of the application modes in the model. Select the model's application mode from the **Merge application mode with** list. The resulting application mode then includes the physics settings for the component.

If you want the component application mode to exist as a new application mode, select **None** in the **Merge application mode with** list. If you do not want to merge the application modes, you must make sure that the application mode names are unique. The program suggests unique names for the dependent variables and the application mode in the corresponding edit fields. If there are no application modes to merge with, or if **None** is selected, COMSOL Multiphysics adds a new application mode to the model. This application mode is only active in the domains belonging to the component.

When you have application modes in the model that the software does not merge with application modes from the component, they are active only in the domains that belonged to the model before the merge.

Because of geometry changes the merge does not include the mesh and the solution. Also, coupling variables that have other destinations than global are not merged.

For an example of how to work with components and merging them into a model, see "Heat-Sink Experiments Using the Component Library" on page 289 in the *COMSOL Multiphysics Model Library.*

13

Using Assemblies

This chapter explains how to model using assemblies where the geometry consists of separate parts. There are special considerations and possibilities during meshing as well as setting up the boundary conditions for an assembly model.

Creating Assemblies

The Analyzed Geometry—Assembly vs. Composite Geometry

When you leave Draw mode COMSOL Multiphysics analyzes the geometry to find all subdomains, boundaries, edges, and points in the geometry. When doing this COMSOL Multiphysics creates an *analyzed geometry* that you apply the physics on. You can choose between two types of analyzed geometries:

- A *composite geometry.* The analyzed geometry object that the software creates is a single object (that typically consists of several subdomains), which effectively is the same object you would get if you select all objects in Draw mode and create the union of them. This is the default type of analyzed geometries, for which COMSOL Multiphysics automatically connects the geometry, mesh, and physics at the interior interfaces.
- An *assembly.* In this case you get an analyzed geometry that consists of several parts (disjoint composite geometries), each object in Draw mode becoming a part of the assembly. For an assembly, you must manually connect the geometry, mesh, and physics at the part interfaces.

If there are no problems using a composite geometry this is usually to prefer, because it automatically provides continuity in the physical fields at part interfaces and material discontinuities as well as accurate solutions and conforming mesh elements and nodes at part interfaces. There are some cases, however, when an assembly is the best choice:

- Sometimes the software fails to create a composite geometry. In those cases you can choose to use an assembly. It is a simpler operation to create an assembly, and it often succeeds even if creating a composite geometry fails.
- Each part in an assembly gets separate meshes. This can make it easier to mesh a complicated geometry, where the mesh generator creates at too dense mesh or even fails to create a mesh for the composite geometry. Bear in mind that having separate meshes means that the meshes may not be compatible at the two sides of a border between two parts. This can give you a less accurate solution.
- Assemblies make it possible to apply boundary conditions where the dependent variable is discontinuous at the interface between two parts. One example is a contact resistance boundary condition for electric currents where the electric potential is discontinuous.

To choose between using a composite geometry and using an assembly, toggle the **Use Assembly** menu item on the **Draw** menu.

Creating Pairs

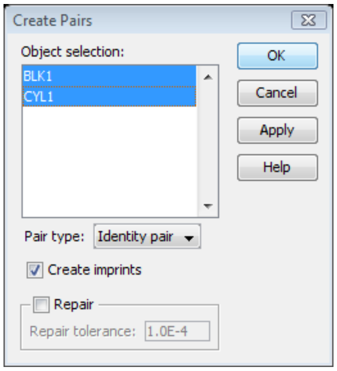
In order for the physics application modes to be able to connect the physics between the parts in the assembly you need to create *identity pairs*. The physics application modes use the pairs when defining their equations. An identity pair consists of two sets of domains: one set with domains from one part and a second set with domains in the other part. The domains in the pairs are those which are in contact with each other. Each pair consists of boundaries, edges, or points, but never a mixture of domain types. For more details about identity pairs, see the section “Identity Pairs” on page 356.

When you create the pairs, you can choose to make *imprints* of one geometry in another. If the boundaries of the two parts which are in contact with each other are of different sizes, COMSOL Multiphysics can create an imprint of the smaller boundary on the larger boundary. In 2D this inserts points on the boundary, and in 3D this creates a curve on the boundary. The advantage of making imprints is that the two sides of the pair match. When you mesh the parts, the meshes match along the border of the pair of boundaries. This provides a more accurate solution than in the case without the imprint, where the meshes do not match.

To create the pairs either use the **Create Pairs** dialog box on the **Draw** menu or the **Create Pairs and Imprints** button and the **Create Pairs** button on the Draw toolbar.

EXAMPLE OF CREATING PAIRS

This example uses a cylinder placed on top of a cube. To create a pair between them open the **Create Pairs** dialog box in the **Draw** menu.


Create Pairs

Object selection:
BLK1
CYL1

Pair type: Identity pair

[✓] Create imprints

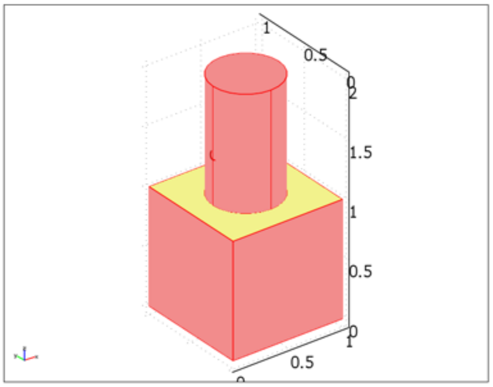
[ ] Repair

Repair tolerance: 1.0E-4

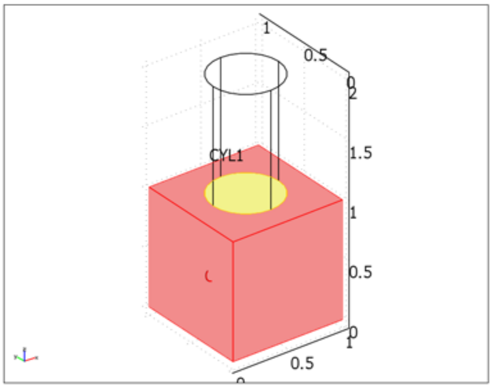
The **Create imprints** check box controls whether to create imprints or not. In the list select which objects to create pairs between. In this case there are only two objects, but when there are more than two, the software tries to create pairs between all pairs of the selected objects. For each pair of objects that touch each other the software creates identity pairs, usually one boundary pair, one edge pair, and one point pair.

If you create pairs between the cube and the cylinder without imprints, the rectangular surface of the cube and the circular end of the cylinder become an identity pair.

In the following figure the boundaries that belong to the boundary pair have an orange color.



If you instead create pairs and imprints, the software creates a circular curve on the cube's surface. This circular surface on the cube and the cylinder surface become a pair. In the following figure you can see this circle.



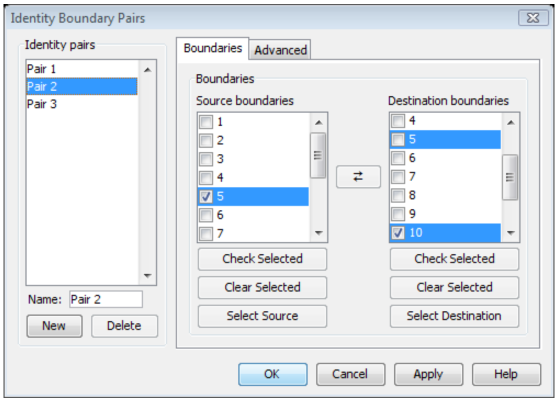
If your license includes the Structural Mechanics Module or the MEMS Module you can also create *contact pairs* using the **Create Pairs** dialog box. To do this select **Contact pair** in the **Pair type** list. This list is absent in the dialog box if you do not have any of these two modules. Contact pairs are used when modeling structural contact. You can read about this topic in the *Structural Mechanics Module User's Guide*.

Identity Pairs

When making models with assemblies you need to define *identity pairs*, which define where the parts of the assemblies are connected. The application modes use the pairs to put constraints on the equations such that the solution becomes continuous across the border between the parts. An identity pair consists of two sets of domains, either two sets of boundaries, two sets of edges, or two sets of points. Such pairs are called identity boundary pairs, identity edge pairs, and identity point pairs, respectively. The two sets of domains are the *source domains* and the *destination domains*. The application modes put a constraint on the dependent variables on the destination domains, forcing them to be equal to the value on the source domains. When the source and destination do not match each other completely you obtain the best numerical result if you let the destination be the smaller of the two. When creating pairs either using the **Create Pairs** dialog box or any of the **Create Pairs and Imprints** button or the **Create Pairs** button on the Draw toolbar, the smaller side always becomes the destination. The easiest way to create the identity pairs is to use the dialog box or the toolbar buttons. For details see the section "Creating Pairs" on page 353; it describes how to manually define the pairs, which becomes necessary if the automatic generation fails, or if you need to adjust the pairs which the automatic generation created.

The Identity Pair Dialog Box

To define identity pairs, choose any of the menu items on the **Physics>Identity Pairs** menu. This opens the dialog box for the identity pairs of the selected domain type, for example, the **Identity Boundary Pairs** dialog box.



Each pair has a name. The application modes use this name to refer to the pair. The names must be unique for each domain level, but pairs on different domain levels can have the same name. For example, two boundary pairs cannot have the name “Pair 1,” while there can be both a boundary pair and a point pair called “Pair 1.”

The two domain lists show the source and destination domains of the pair selected in the list to the left. The check boxes beside the domain numbers indicate which domains belong to the source and which to the destination.

Clicking the buttons **Check Selected** below the lists selects the check boxes of the domains highlighted in the list. This is equivalent to selecting the individual check boxes and is a quick way to select multiple check boxes. Clicking the button **Clear Selected** similarly clears the check boxes of the selected domains.

Use the buttons **Select Source** and **Select Destination** to select the source and destination domains in the main window and in the selection lists.

Clicking the arrow button between the selection lists interchanges the source and destination domains.

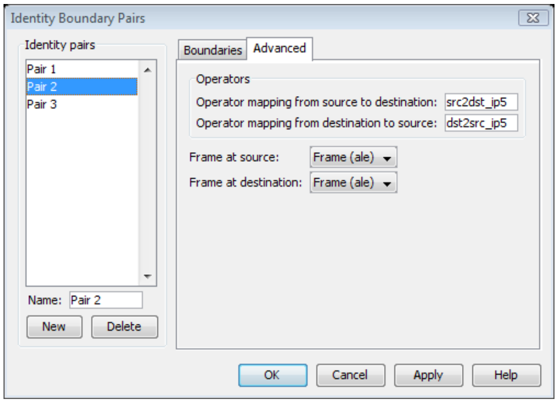
TIPS FOR MODELING USING IDENTITY PAIRS

Note these aspects of the modeling procedure using identity pairs:

- When adding identity pairs, use the domain with the coarser mesh as the source domain and the domain with a finer mesh as the destination domain.
- When using identity pairs to connect boundaries or edges that do not have equivalent meshes, use the sparse null-space function, particularly in 3D. Normally the automatic selection takes care of this. To specify the sparse null-space function, open the **Solver Parameters** dialog box, click the **Advanced** tab, and select **Sparse** in the **Null-space function** list.

COUPLING OPERATORS

On the **Advanced** page in the **Identity Pair** dialog box you can define the names of the pair's coupling operators. A *coupling operator* evaluates its argument on one side of the pair and makes the result available on the other side.



Identity Boundary Pairs

Identity pairs: Pair 1, Pair 2, Pair 3

Name: Pair 2

Operators

Operator mapping from source to destination: src2dst_ip5

Operator mapping from destination to source: dst2src_ip5

Frame at source: Frame (ale)

Frame at destination: Frame (ale)

In the previous figure you can see two operators: `src2dst_ip5`, mapping from the source of the pair to the destination, and `dst2src_ip5`, mapping in the other direction. For example, if `u` is a variable on the source side then you can use the expression `src2dst_ip5(u)` on the destination side.

When creating a pair COMSOL Multiphysics automatically defines the operators and give them a name. The names have to be unique within the whole model.

The application modes use the operators to define the constraint which forces the dependent variables to be continuous across the border between the parts of the assembly. For example, if `u` is a dependent variable, the application modes defines the constraint `src2dst_ip5(u) = u` on the destination side of the pair.

You can also use the operators in logical expressions. The operator name without an argument evaluates to true in the points which have a corresponding source point in the same position. As an example consider the cube and cylinder in the figure below. Assume that the rectangle on top of the cube is the source of a pair and that the bottom cylinder surface is the destination. Further assume that the operator `dst2src` maps from the source to the destination. Then you can use the expression `if(dst2src,1,2)` on the source side (the cube). It evaluates to 1 in the points on the surface touching the cylinder and to 2 in the other points.

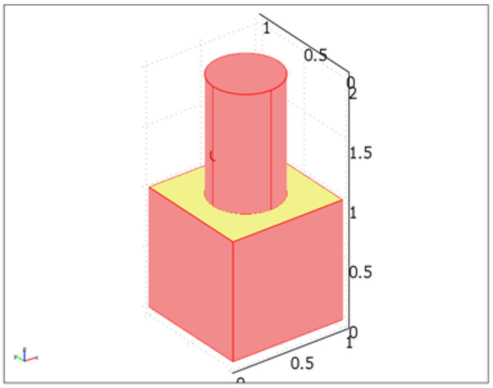


Figure 13-1: A cube and a cylinder which are two parts of an assembly. The cube's upper surface is the source boundary of an identity pair and the cylinder's bottom surface is the destination boundary of the pair.

USING FRAMES FOR PAIRS

If there is more than one frame in a model you can choose on which frame to evaluate the expressions when using coupling operators. If you choose two different frames for the source and destination, you can simulate that the source and destination are translated or rotated relative to each other.

The panel where you can select frames does not appear when there are no extra frames in the model.

Selection Colors for Pairs

When you select a domain it is normally highlighted in red. In 3D selected domain can also be blue and green for confirmed selections (see the section "Object Selection Methods in 3D" on page 127). In addition, pairs have their own selection colors.

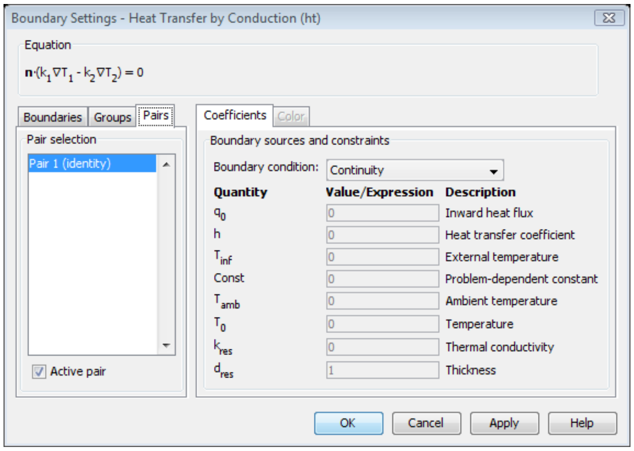
In Draw mode the domains belonging to a pair have an orange color. This is true whether they are selected or not.

In the Subdomain, Boundary, Edge, and Point selection modes the pair domains have three possible selection colors. When you select a source domain it becomes highlighted in cyan. A destination domain which you select becomes highlighted in magenta. If a domain is a source domain to one pair and a destination domain to another pair it is highlighted in orange. These specific selection colors for pairs override the red selection colors but not the green and blue colors for confirmed selections.

You can deactivate the pair colors and use only the red selection color. To do so open the **Preferences** dialog box, and on the **Visualization** tab clear the **Pair colors** check box. This also turns off the orange color for the pairs in Draw mode.

Specifying Physics Settings on Pairs

The application modes use the identity pairs described in the previous chapter to define the boundary conditions at the border between the domains in the pair. The default boundary condition ensures continuity of the dependent variables across this border, but you can specify other boundary conditions. To do this, use the **Boundary Settings**, **Edge Settings**, and **Point Settings** dialog boxes. Whenever there are pairs in a model, a **Pairs** page appears in the dialog box (see the following figure).


Boundary Settings - Heat Transfer by Conduction (ht)

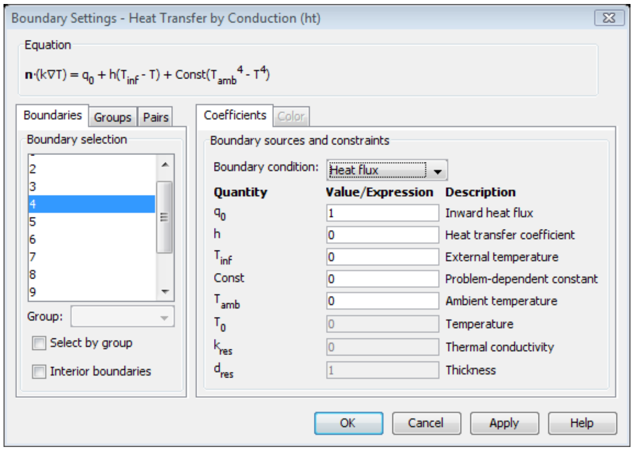
Equation

$\mathbf{n}\cdot(k_1\nabla T_1 - k_2\nabla T_2) = 0$

Boundary condition: Continuity

Quantity	Value/Expression	Description
q_0	0	Inward heat flux
h	0	Heat transfer coefficient
T_{inf}	0	External temperature
Const	0	Problem-dependent constant
T_{amb}	0	Ambient temperature
T_0	0	Temperature
k_{res}	0	Thermal conductivity
d_{res}	1	Thickness

The boundary condition that you apply to a pair is active everywhere where the two sides of the pair are in contact. But you can also set boundary conditions on the individual boundaries of the pair. Do so on the **Boundaries** page. These settings are active everywhere where the domains of the pair are not in contact with each other. To clarify how this works, consider the example of a cube with a cylinder on top of it in Figure 13-1. Assume that the cube's upper surface is Boundary 4 and the cylinder's bottom surface is Boundary 9. The boundary condition on the pair applies to the circular area where the two surfaces are in contact. You specify the boundary condition for the rest of Boundary 4 on the **Boundaries** page. In the following figure this boundary condition is a heat flux boundary condition.


Boundary Settings - Heat Transfer by Conduction (ht)

Equation

$\mathbf{n}\cdot(k\nabla T) = q_0 + h(T_{inf} - T) + Const(T_{amb}^4 - T^4)$

Boundary selection: 2, 3, 4, 5, 6, 7, 8, 9 (4 selected)

Group:

Select by group

Interior boundaries

Boundary sources and constraints

Boundary condition: Heat flux

Quantity	Value/Expression	Description
q_0	1	Inward heat flux
h	0	Heat transfer coefficient
T_{inf}	0	External temperature
Const	0	Problem-dependent constant
T_{amb}	0	Ambient temperature
T_0	0	Temperature
k_{res}	0	Thermal conductivity
d_{res}	1	Thickness

If you specify a boundary condition at Boundary 9 in this list, this boundary condition has no effect. This is because all of Boundary 9 is in contact with Boundary 4 and therefore the boundary condition of the pair applies everywhere on this boundary.

Controlling the Use of Pairs

On the **Pair** page you find the **Active pair** check box. Clearing this check box makes the pair inactive for this application mode. In this case the application mode does not use the boundary condition on the pair but instead uses the boundary conditions on the individual boundaries that you specify on the **Boundaries** page. When a pair is not active the application mode does not ensure that the dependent variables are continuous across the border between the parts. Instead the boundaries on the two sides of the pair behave like exterior boundaries, and the parts are completely insulated from each other.

Pairs also become inactive if you have deactivated the application mode in one part of the geometry such that one or both sides of the pair belongs to the inactive part. In that case the active side of the pair is an exterior boundary, and the settings on the **Boundaries** page apply.

Edge and Point Pairs

You specify the settings for edge and point pairs in the same ways as for boundary pairs. But most application modes do not use the edge and point pairs. The application modes that define their equations on subdomain level use boundary pairs only. The reason is that because the setting on the boundary pair ensures that the dependent variables are continuous between the boundaries, this also ensures that they are continuous between the edges and points adjacent to the boundaries in contact. Putting constraints on the edges and points is therefore unnecessary.

The application modes which define their equations on the boundary level use the pairs on edges in 3D or on points in 2D. The dependent variables of these application modes need a constraint on the edges (or points) to be continuous.

Similarly, the application modes that define their equations on the edge level use the point pairs.

Slit Boundary Conditions

On the borders between pairs in an assembly, you can specify special "slit boundary conditions," which represent a discontinuity in the field due to, for example, contact resistance. The following application modes include a boundary condition of this type:

TABLE 13-1: SLIT BOUNDARY CONDITIONS

APPLICATION MODE	BOUNDARY CONDITION	PHYSICAL QUANTITY
Conductive Media DC	Contact resistance	Electric conductivity
Electrostatics	Thin highly capacitive layer	Permittivity
Diffusion, Convection and Diffusion, Electrokinetic Flow	Thin boundary layer	Diffusion coefficient
Heat Transfer by Conduction, Convection and Conduction, General Heat Transfer	Thin thermally resistive layer	Thermal conductivity
Magnetostatics, No Currents	Thin low permeability gap	Permeability

These boundary conditions are available in the **Boundary Settings** dialog box by clicking the **Pairs** tab and selecting the pairs where you want to apply the boundary condition. In addition to the physical quantity listed in Table 13-1, you must also enter the thickness of the layer or gap. For an example model, see "Thin-Film Resistance Model" on page 368.

Note: The General Heat Transfer application mode is only available in the Heat Transfer Module, and the Magnetostatics, No Currents application mode is only available in the AC/DC Module. The Electrokinetic Flow application mode is only available in the MEMS Module and the Chemical Engineering Module.

Meshing Assemblies

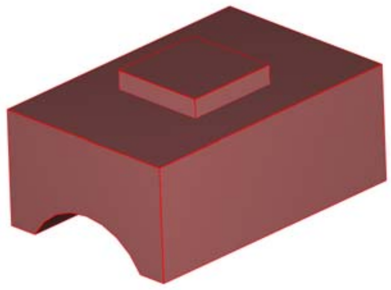
A geometry object that is to be meshed is classified as either a *composite geometry* or an *assembly*. For more information on the two geometry types see "The Analyzed Geometry—Assembly vs. Composite Geometry" on page 352.

When meshing an assembly, the mesh generator creates meshes for the individual parts of the assembly independently of each other. This means that a geometry object of assembly type often gives you more freedom when determining the meshing strategy than the corresponding composite geometry. Furthermore, in most cases, an assembly generates a mesh with fewer elements than the corresponding composite geometry.

Whether you create an assembly with or without imprints also affects the meshing process. An assembly created without imprints gives you even more freedom when determining the meshing strategy than the corresponding assembly with imprints.

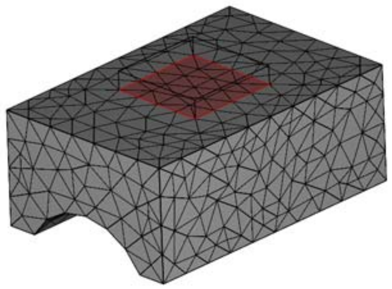
Example—Meshing a Composite Geometry or an Assembly

To show the different meshing possibilities for a geometry object of composite geometry type and assembly type (with and without imprints), this example uses the following geometry object consisting of two subdomains.

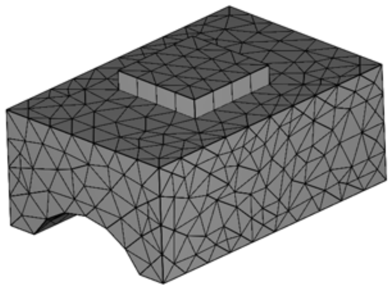


COMPOSITE GEOMETRY

If the geometry object is a composite geometry, there is only one face connecting the two subdomains. This face, common to both subdomains, is called an *interior boundary* of the geometry object. Due to topology reasons of the lower subdomain (that is, the connectivity of the faces about the subdomain) you must use the free mesher when meshing the lower subdomain. This means that the interior boundary face contains triangular boundary elements (see the following figure).

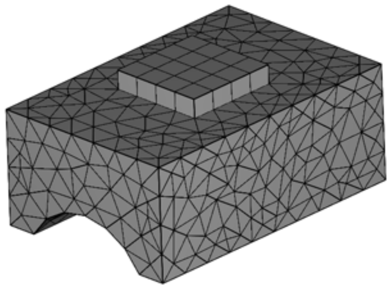


When meshing the upper subdomain you can use the free mesher or the swept mesher. However, if you use the swept mesher for this subdomain the interior boundary face must be used as the source face or the target face for the sweep. Hence, it is not possible to create a swept hexahedral mesh on the upper subdomain. The next figure shows a swept prism mesh on the upper subdomain.



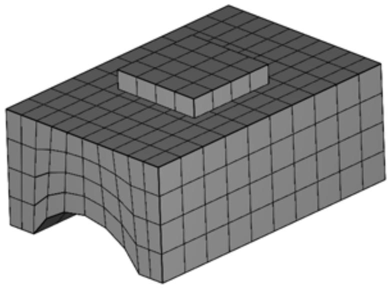
ASSEMBLY WITH IMPRINTS

If you model the geometry object as an assembly with imprints, COMSOL Multiphysics creates two separate faces of the same shape, belonging to each part, at the border between the parts. The meshes for these faces are independent of each other. Due to the imprint on the lower part, it is still only possible to create a free mesh on this part. However, because you can create a mesh for the upper part independently from the lower part you can create a swept hexahedral mesh on the upper subdomain.



ASSEMBLY WITHOUT IMPRINTS

If you form an assembly without imprints you can freely mesh the two parts (subdomains) independently of each other. Because there is no imprint of the upper part on the lower part, it is now possible to create a swept mesh on the lower part as well. In the next figure, the mesh generator creates a swept hexahedral mesh on both parts of the assembly.



Thin-Film Resistance Model

When modeling transport by diffusion or conduction in thin layers, large differences in dimensions of the different subdomains are common. If the model has a sandwich structure, you can replace the thinnest layers with a thin-layer approximation, provided that the difference in thickness is large.

Model Definition

This study explains the principle of the thin-layer approximation in direct current conduction problems. A comparison of a structure with three subdomains to a simplified model that replaces the domain in the middle with a thin-layer approximation shows the benefit of this approach (see Figure 13-2).

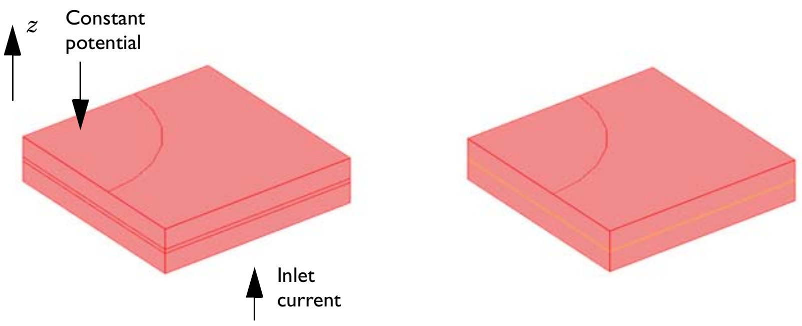


Figure 13-2: Exact domain description (left) and approximation (right). The current flows from the base plate to the circular plate on the upper surface of the device.

Equation 13-1 below describes the current balance in all three subdomains in the real sandwich structure:

$$\nabla \cdot (-\sigma \nabla V) = 0 \qquad (13\text{-}1)$$

In this equation, σ represents the conductivity and V the electric potential. In this case, there is a substantial difference in conductivity between the thin and thicker layers of the structure. The boundary conditions include a current inlet in the base plate of the device and a constant potential at the upper circular boundary (see Figure 13-2). All other boundaries are insulated.

The simplified model is based on the assumption that the components of the current density vector in the x and y directions are small and that the dominating transport through the thin structure is obtained in the *z* direction. For the middle layer, this implies that you can approximate Equation 13-1 by the one-dimensional equation

$$-\sigma\frac{d^2V}{dz^2} = 0 \tag{13-2}$$

It is possible to solve this equation analytically if the potential is given at the lower and upper surfaces of the middle layer:

$$V_{\delta = 0} = V_1 \tag{13-3}$$

$$V_{\delta = \delta_1} = V_2 \tag{13-4}$$

You can integrate Equation 13-2 analytically to give:

$$V = az + b \tag{13-5}$$

where *a* and *b* are integration constants. If you arbitrarily place $z = 0$ at the lower boundary of the middle layer, you get the constants *a* and *b* from the boundary conditions in Equation 13-3 and Equation 13-4:

$$V_1 = b \tag{13-6}$$

$$V_2 = a\delta + b \tag{13-7}$$

This gives:

$$b = V_1 \tag{13-8}$$

$$a = \frac{V_2 - V_1}{\delta} \tag{13-9}$$

The resulting equation for the potential is thus

$$V = \left(\frac{V_2 - V_1}{\delta}\right)z + V_1 \tag{13-10}$$

The current density is defined as

$$J_z = -\sigma\frac{dV}{dz} \tag{13-11}$$

Combining Equation 13-10 and Equation 13-11 gives

$$J_z = -\sigma\left(\frac{V_2 - V_1}{\delta}\right) \qquad (13\text{-}12)$$

In the thin-film approximation the potential is discontinuous at the film boundary. Using an assembly it is possible to model such a potential. The Conductive Media DC application mode has a contact resistance boundary condition available at pairs, which uses the thin-film approximation.

It is also possible to derive the expression for the current density in Equation 13-12 by approximating the gradient using the potential difference over the thin layer. This example includes the previous tedious derivation to show that this is exactly what you obtain from the solution of Equation 13-2.

The approximation presented in this example is not limited to direct current problems: You can also use it for modeling of diffusion, heat conduction, flow through porous media using Darcy's law, and other types of physics that the divergence of a gradient flux describes.

In general, the application of this simplification is appropriate in cases where the differences in thickness are so large that the mesh generator cannot even mesh the domain. In some cases, the mesh generator might be able to mesh the domain but then creates a very large number of elements.

Results

Figure 13-3 shows a comparison between the exact solution of the problem using three conductive layers and the thin-film approximation. The comparison reveals an excellent agreement in the potential and current distribution despite that the middle film in this study is relatively thick. The approximation becomes even more accurate as the film thickness between the upper and lower subdomain decreases.

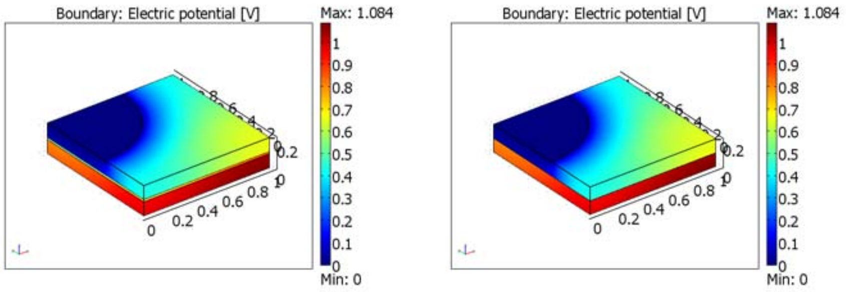

Boundary: Electric potential [V]
Max: 1.084
Min: 0
Boundary: Electric potential [V]
Max: 1.084
Min: 0

Figure 13-3: Potential distribution in the modeled device. The value of the potential loss over the device at a current of 0.3 A is almost identical in the two models.

Figure 13-4 shows a cross-section plot of the potential through the structure's center for the full model and for the approximation. The plots show the excellent agreement obtained between the two models.

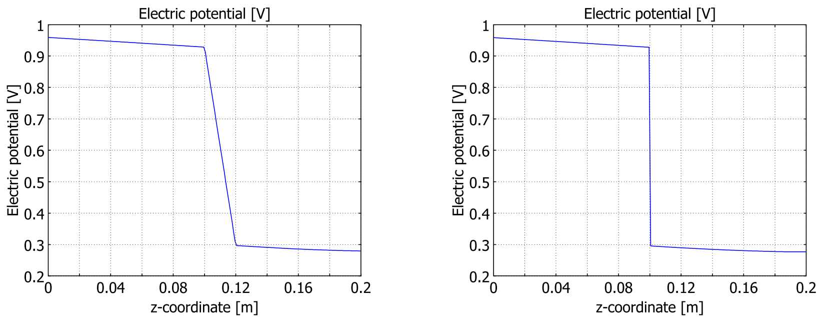


Figure 13-4: Potential distribution along the z direction in the middle of the device.

Model Library path: `COMSOL_Multiphysics/Electromagnetics/thin_film_resistance`

Modeling Using the Graphical User Interface

MODEL NAVIGATOR

1 Start COMSOL Multiphysics.Under the **New** tab select **3D** in the **Space dimension** list.

2 From the list of application modes, select **Conductive Media DC**, which you find under **COMSOL Multiphysics>Electromagnetics**.

3 Click **OK**.

GEOMETRY MODELING

1 Draw the bottom block by selecting **Draw>Block.** Set the thickness of the block to `0.1` in the z direction (this is set in the **Length** frame) and leave the other settings as they are. Click **OK**.

2 Draw the top block in the same way except for the **Axis base point**, which should be at $x = 0$, $y = 0$, and $z = 0.1$.

3 Click the **Zoom Extents** button on the Main toolbar to get a better view of the geometry.

4 Now go to **Draw>Work-Plane Settings** and click **OK** to create a work plane with the default settings. A work plane is created under the new **Geom2** tab, while the original 3D geometry remains under the **Geom1** tab.

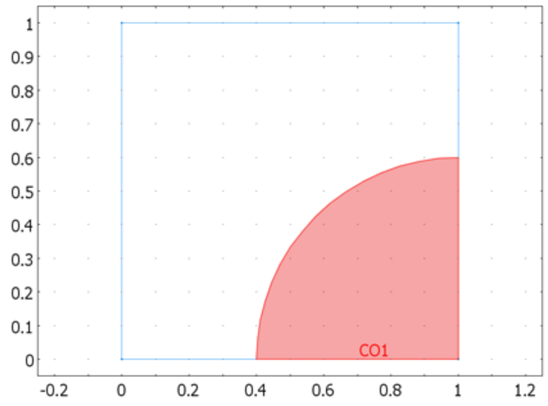
5 Now go to the work plane by clicking on the **Geom2** tab. Click **Zoom Extents** to see the outlines of the 3D geometry from **Geom1**.

6 Draw a circle centered at (1, 0) with a radius of 0.6. This is done by selecting **Draw>Specify Objects>Circle**. In the **Size** area set the **Radius** to `0.6`, and in the **Position** area set **Base** to **Center** and the coordinate **x** to `1` and **y** to `0`.

7 Draw a square by clicking the **Rectangle/Square** button from the Draw toolbar on the left and click on the point (0, 1) and then (1, 0).

8 Create an object by finding the intersection of the square and the circle. This is done by selecting both objects (press Ctrl+A) and then clicking on the **Intersection** tool

on the toolbar on the left. The 2D geometry in **Geom2** should now look as in the figure below:



The work plane.

9 Now embed the quarter-circle in the 3D geometry by selecting **Draw>Embed** and clicking **OK**.

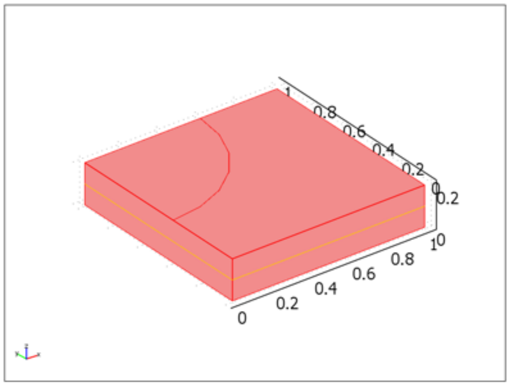
10 Move the flat quarter-circle surface to the top corner of the structure by pressing Ctrl+X followed by Ctrl+V. For the displacements, type `-1`, `1`, and `0.2` in the **x**, **y**, and **z** edit fields, respectively, then click **OK**.

11 Rotate the quarter circle by selecting it and clicking on the **Rotate** tool on the toolbar on the left. Type `180` in the **Rotation angle** edit field (the rotation angle α in degrees) and specify the point on the rotation axis to be (0, 1, 0) in the **Point on rotation axis** area. Click **OK**.

12 Select the embedded object `EMB1` and the upper block `BLK2` and click the **Coerce to Solid** button to combine the two objects to a single object.

13 Click the **Create Pairs** button on the Draw toolbar to create the pair joining the two objects.

The final geometry should look as in the figure below:



The final 3D geometry.

PHYSICS SETTINGS

Subdomain Settings

1 Select **Physics>Subdomain Settings**.

2 Select all subdomains.

3 In the σ edit field (**Electric conductivity**) type `1`. Click **OK**.

Boundary Conditions

1 Select **Physics>Boundary Settings** and set the following boundary conditions:

SETTINGS	BOUNDARY 3	BOUNDARY 11	ALL OTHERS
Boundary condition	Inward current flow	Ground	Electric insulation
J_n	`0.3`		

2 Click the **Pairs** tab and select **Pair 1**. From the **Boundary condition** list, select **Contact resistance**. In the σ edit field, enter the electric conductivity `1e-2` and in the **d** edit field enter the thickness `0.02`. Click **OK**.

COMPUTING THE SOLUTION

Click the **Solve** button on the Main toolbar to mesh and solve the model.

POSTPROCESSING AND VISUALIZATION

Reproduce the right panel of Figure 13-3 by following these instructions:

1 Click the **Plot Parameters** button on the Main toolbar.

2 On the **General** page, clear the **Slice** check box and select the **Boundary** check box.

3 Click the **Boundary** tab.

4 From the **Predefined quantities** list, select **Conductive Media DC (dc)>Electric potential**.

5 Clear the **Smooth** check box.

6 Click **OK**.

To generate the plot in the right panel of Figure 13-4, proceed with the following steps:

1 From the **Postprocessing** menu, select **Cross-Section Plot Parameters**.

2 On the **Line/Extrusion** page, verify that the selection in the **Predefined quantities** list in the **y-axis data** area is **Potential**.

3 In the **x-axis data** area, click first the lower option button and then the **Expression** button.

4 In the **X-Axis Data** dialog box, type **z** in the **Expression** edit field. Click **OK**.

5 In the **Cross-section line data** area, enter the following settings:

x0	0.5	x1	0.5
y0	0.5	y1	0.5
z0	0	z1	0.2

6 Click **OK** to close the **Cross-Section Plot Parameters** dialog box and generate the plot.

Comparing the Thin-Film Approximation with the Full 3D Model

You can easily reproduce the full 3D model that served as a reference model for comparison in this example. To do so, move the upper block, CO1, a distance 0.02 in the z direction, then add a thin block with the height 0.02 in between the top and bottom block. Here it is not necessary to use an assembly, so you can clear **Use Assembly** in the **Draw** menu. Set the conductivity to 0.01 in this thin middle subdomain (leave it at 1 in the top and bottom subdomains). Let the boundary conditions for the top and bottom of the thin layer be inactive, which means that the solution is continuous across the interfaces. Let the outer boundary conditions be the same as they were in the above description (that is, electric insulation), and compute the solution for this model.

It should now be apparent that the two solutions show good agreement and that the first approach consumes significantly less time and memory.

Using Identity Conditions

To connect the fields of physics across different geometries, you can use identity conditions that make the physical quantities equal across the different parts. The identity boundary conditions and identity conditions on other domain types define a constraint that makes two quantities equal on two separate (but usually equally-shaped) domains in two different geometries where you do not need a coordinate transformation between the source and destination domains. That is, the source and the destination domains must lie in the same place in the coordinate space.

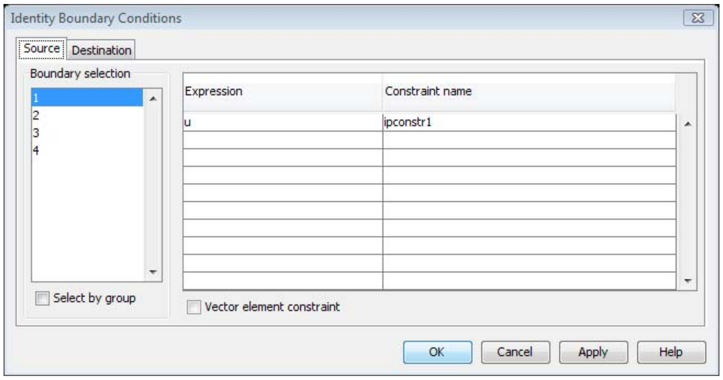
Note: Using an *assembly* and *identity pairs*, you can work with a geometry that consists of several parts without using multiple geometries and identity conditions.

Creating Identity Conditions

When you do not need a transformation, you specify a identity condition in the same way as you specify a periodic condition:

1. Select the source geometry for the coupling variable.
2. On the **Options** menu point to **Identity Conditions**, then click on the domain type: **Identity Subdomain Conditions**, **Identity Boundary Conditions**, **Identity Edge Conditions** (3D only), or **Identity Point Conditions**.
3. In the dialog box for the identity conditions, start by specifying the source geometry on the **Source** page. Enter the expression that you wish to use in the identity coupling for the source. Refer to each constraint in the identity coupling by a name. A default name appears for the constraint that implements the identity coupling. To

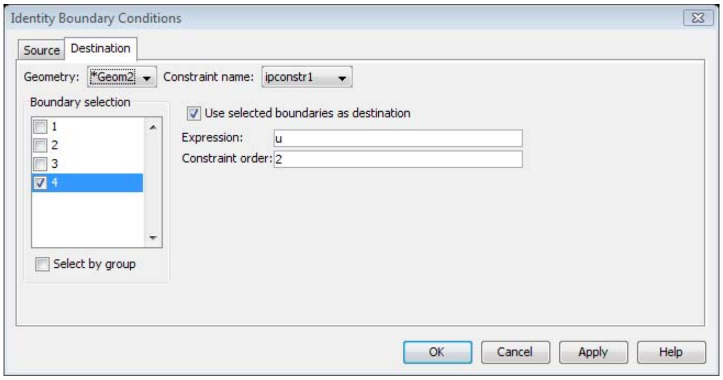
change the default name of the constraint while working on the source page, type another name in the **Constraint name** column.



Expression	Constraint name
u	ipconstr1

Figure 13-5: The Source page in the Identity Boundary Conditions dialog box.

4 Click the **Destination** tab to define the destination domains. Select the destination geometry in the **Geometry** list. Then select the constraint that you want to specify in the **Constraint name** list. Finally specify the expression that becomes equal to the expression on the source domain for the corresponding constraint.



Geometry: *Geom2
Constraint name: ipconstr1
Boundary selection: [✓ 4]
[✓] Use selected boundaries as destination
Expression: u
Constraint order: 2

Figure 13-6: The Destination page in the Identity Boundary Conditions dialog box.

5 Click **OK** when you have defined all the identity conditions for the domain type that you are working with.

See the model example on the following pages for a description of how to combine two geometries with brick and prism meshes using an identity boundary condition.

MODELING PROCEDURE USING IDENTITY CONDITIONS

Note these aspects of the modeling procedure using identity conditions:

- When adding identity conditions, use the domain with the coarser mesh as the source domain and the domain with a finer mesh as the destination domain.
- For an identity boundary condition, use a Neumann boundary condition for the boundaries on both the source and destination domains. The identity boundary conditions do not work if you use a Dirichlet boundary condition. You specify the boundary conditions in the **Boundary Settings** dialog box.
- If the source domain is larger than the destination domain and you would like to use a different type of boundary condition on the part of that boundary that is not connected to the destination boundary in the other domain, you can introduce a smaller domain that matches the destination domain. This might, however, limit the possibilities to use a mapped 2D mesh and then also a 3D mesh using brick elements. In that case, consider using an unstructured triangular 2D mesh and mesh extrusion to form a 3D mesh using prism elements if you want to avoid an unstructured tetrahedral mesh.
- When using identity-coupling variables to connect boundaries or edges that do not have equivalent meshes, use the sparse null-space function, particularly in 3D. Normally the automatic selection takes care of this. To specify the sparse null-space function, open the **Solver Parameters** dialog box, click the **Advanced** tab, and select **Sparse** in the **Null-space function** list.
- Use the same name for the dependent variables and the application modes that represent the same physics in different geometries. This makes it convenient to postprocess the results in the entire geometry. Select all the geometries in the **Geometries to use** list in the **Plot Parameters** dialog box to plot the results in all parts of the model. The default is that you have unique names, but it is possible to edit the names of the application mode and the dependent variable to be the same as long as the application modes belong to different geometries. For an example of how to do this, see “Thin-Layer Diffusion Model” on page 380.

IMPLEMENTATION OF THE IDENTITY CONDITIONS

COMSOL Multiphysics implements an identity condition as a special case of the extrusion coupling variable. In particular, it adds a variable name equal to the constraint name on the destination domains with a value equal to the constrained expression. The constraint works as an *ideal constraint*, that is, the derivatives of the constraint with respect to the solution help form the reaction forces. If you need more freedom in specifying how the reaction forces are applied, use the extrusion coupling variable directly and specify the reaction force and constraint as weak contributions.

Note: It is normally much easier to work with an *assembly* that consists of two or more geometrical parts. You can then use identity pairs and interactive meshing to create a model where the physics is connected and where you can use different types of meshes in each part.

MODELING PROCEDURE FOR MULTIGEOMETRY LINKING

The following list shows the main steps you must take to make a model with a continuous field that you link between two or more geometries:

1. Create the full geometry using separate geometries (parts) in the COMSOL Multiphysics model. Use the **View All 3D Geometries/View All 2D Geometries** button in the Visualization/Selection toolbar or the **View Geometries** dialog box to show other geometries than the current geometry during modeling. This helps to see that the geometries are spatially connected and that the total geometry looks right.
2. Create the meshes for the different geometries. These can be of different kinds, which is not possible using a single geometry. This is the main advantage with this multigeometry approach.
3. Add the physics to the different geometries. It is typically the same type of physics, and the field is continuous across the geometries. For this purpose, use the same name for the field variables (dependent variables) and the application modes. You can then visualize and postprocess the solution on the entire geometry using the same variable names for both dependent variables and application mode variables that use the application mode name as a suffix.

Note: The default names for the application modes and dependent variables are different for each application mode that you add. Make sure to edit these to make all names the same.

---

4. Use identity boundary conditions to link the physics across the different geometries.
5. Compute the solution.
6. For visualization plot the results in all geometries simultaneously. To do so, select all the geometries in the **Geometries to use** list in the **Plot Parameters** dialog box.

Thin-Layer Diffusion Model

The following example shows how to link two separate geometries into a 3D thin-layer diffusion model using identity boundary conditions. It also illustrates the use of different mesh element types; the brick and prism elements provide a large reduction in the number of degrees of freedom (DOFs) for this thin geometry. For more information about mesh elements and meshing options, see "Meshing" on page 285 in the *COMSOL Multiphysics User's Guide*.

In this model you can significantly reduce the number of DOFs—and thus the solution time—by taking advantage of the geometry shape using brick and prism meshes instead of an unstructured tetrahedral mesh. The total model has only about 2300 DOFs using this approach. Using a default tetrahedral mesh on the same geometry leads to a problem with roughly 18,000 DOFs.

The physics in this model is a single-species diffusion. The dependent variable is the concentration, c. All boundaries are insulated except the inlet and the outlet. At the inlet boundary the concentration is c_0. At the outflow boundary (the bottom surface) there is an outward flux of $-r_{\mathrm{surf}} c / c_0$ (COMSOL Multiphysics defines the inward flux as positive), where r_{surf} is the reaction rate at the surface. An effective diffusion coefficient takes the porous material in the thin bottom plate into account. Table 13-2 on page 387 lists all material properties in the model.

The interface condition at the connection between the top and bottom parts is that the concentration c is equal on both sides.

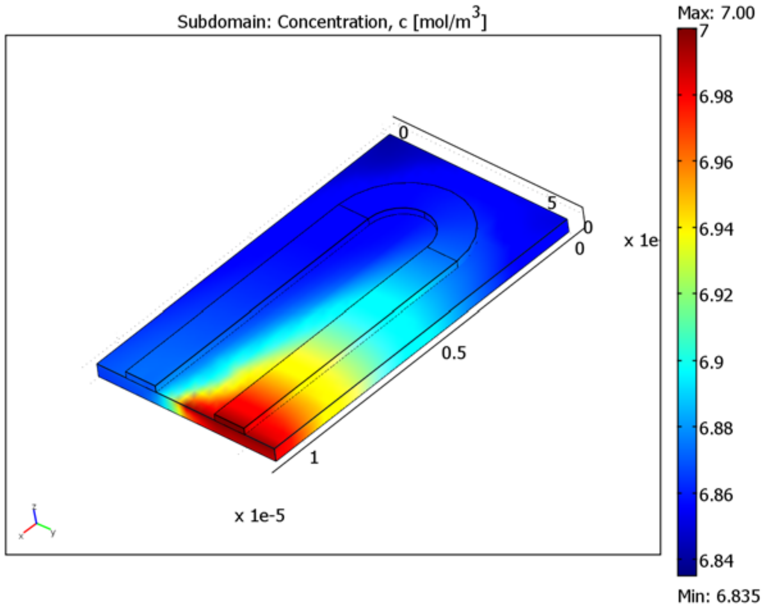


Figure 13-7: Resulting concentration distribution in the full geometry.

Model Library path:
`COMSOL_Multiphysics/Diffusion/thin_layer_diffusion`

Modeling Using the Graphical User Interface

MODEL NAVIGATOR

1. Start COMSOL Multiphysics.
2. In the **Model Navigator** click the **Multiphysics** button.
3. Click the **Add Geometry** button to add a 2D geometry.
4. Click **OK** to accept the default settings and create the first geometry: **Geom1 (2D)**.
5. Repeat Steps 3 and 4 to create a second 2D geometry: **Geom2 (2D)**.
6. Select **Geom1 (2D)** to make it the current geometry.
7. Click **OK**.

GEOMETRY MODELING

Geometry 1 (Base Block Work Plane)

1 Shift-click the **Rectangle/Square** button on the Draw toolbar.

2 In the **Rectangle** dialog box, type `11e-6` in the **Width** edit field and `6e-6` in the **Height** edit field.

3 Click **OK**.

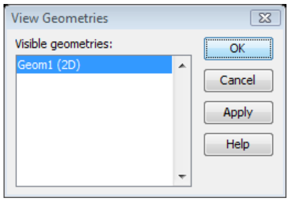
4 Click the **Zoom Extents** button on the Main toolbar.

Geometry 2 (Top Layer Work Plane)

1 Click the **Geom2** tab to switch to the other 2D geometry.

2 From the **Options** menu choose **View Geometries**.

3 In the **View Geometries** dialog box select **Geom1 (2D)**, then click **OK**.



You can also use the **View All 2D Geometries** button in the Visualization/Selection toolbar.

4 Click the **Zoom Extents** button on the Main toolbar.

5 Draw a rectangle with corners at (3e-6, 5e-6) and (1.1e-5, 4e-6).

6 Press Ctrl+C to make a copy of the rectangle, then press Ctrl+V to paste the copy.

7 In the **Paste** dialog box, type `-3e-6` in the **y** edit field, then click **OK**.

8 Draw a centered circle, C1, with its center at (3e-6, 3e-6) and a radius of 2e-6 by first clicking the **Ellipse/Circle (Centered)** button on the Draw toolbar, and then using the right mouse button in the drawing area.

9 Draw another centered circle, C2, with its center at (3e-6, 3e-6) and a radius of 1e-6.

10 Draw a rectangle, R3, with corners at (3e-6, 5e-6) and (5e-6, 1e-6).

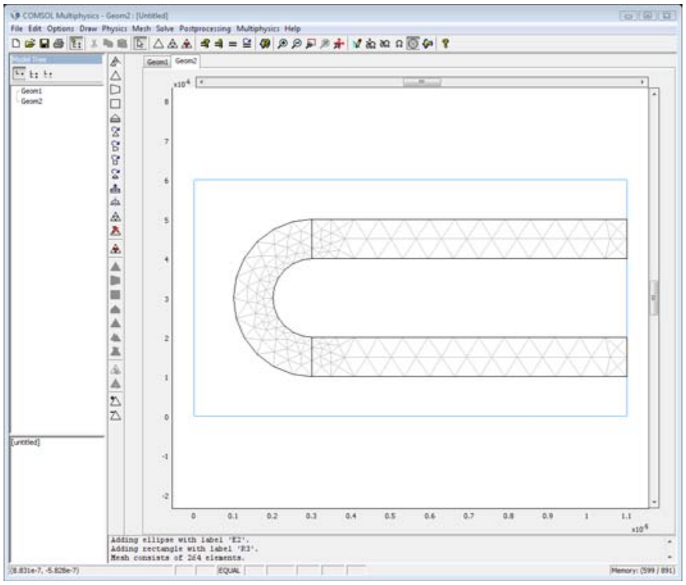
11 From the **Draw** menu choose **Create Composite Object**.

12 In the **Set formula** edit field, type `C1-C2-R3`. Click **OK**.

MESH GENERATION

Geometry 2 (Top Layer Work Plane)

Click the **Initialize Mesh** button to create a triangular mesh for the top U-shaped part.



```
Adding ellipse with label 'E2'.
Adding rectangle with label 'R3'.
Mesh consists of 264 elements.
```

COMSOL Multiphysics - Geom2 : [Untitled]
File Edit Options Draw Physics Mesh Solve Postprocessing Multiphysics Help
Model Tree
Geom1
Geom2
[untitled]
Geom1 Geom2
EQUAL
Memory: (599 / 891)

The triangular mesh for the top layer.

Geometry 1 (Base Block Work Plane)

1 Click the **Geom1** tab.

2 From the **Mesh** menu, choose **Mapped Mesh Parameters**.

3 Click the **Boundary** tab.

4 Select Boundary 1.

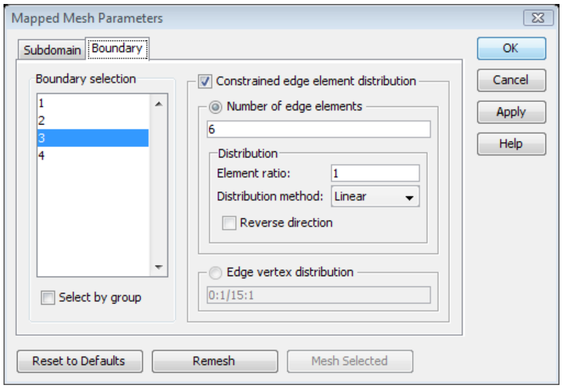
5 Click to select the **Constrained edge element distribution** check box.

6 In the **Number of edge elements** edit field, type 4.

7 Select Boundary 3.

8 Click to select the **Constrained edge element distribution** check box.

9 In the **Number of edge elements** edit field, type 6.


Mapped Mesh Parameters

Subdomain | Boundary

Boundary selection: 1, 2, 3, 4 (3 selected)

☐ Select by group

☑ Constrained edge element distribution

◉ Number of edge elements: 6

Distribution

Element ratio: 1

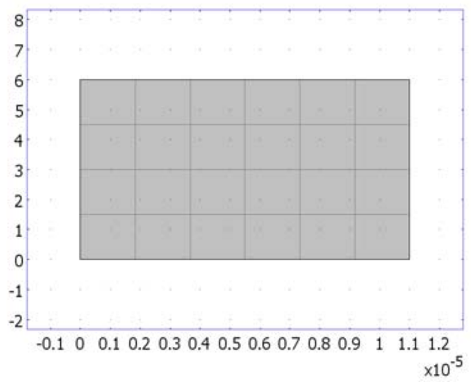
Distribution method: Linear

☐ Reverse direction

○ Edge vertex distribution: 0:1/15:1

10 Click the **Remesh** button. This creates a 4-by-6 quadrilateral mesh.

11 Click **Cancel**.



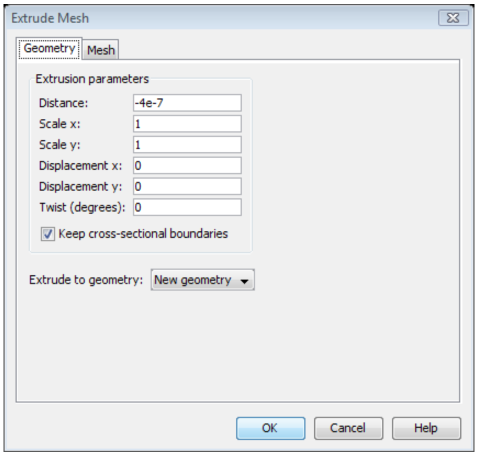
The quadrilateral mesh for the base block work plane.

Geometry 3 (Base Block)

The next step is to extrude the mapped mesh into a 3D brick mesh.

1 From the **Mesh** menu, choose **Extrude Mesh**.

2 In the **Distance** edit field, type `-4e-7`.



Extrusion parameters

Distance: -4e-7
Scale x: 1
Scale y: 1
Displacement x: 0
Displacement y: 0
Twist (degrees): 0
☑ Keep cross-sectional boundaries

Extrude to geometry: New geometry

3 Click **OK** to extrude the quadrilateral mesh 0.4 micrometer in the negative *z* direction. This creates a brick mesh in a new 3D geometry, **Geom3 (3D)**.

Geometry 4 (Top Layer)

1 Click the **Geom2** tab.

2 From the **Mesh** menu, choose **Extrude Mesh**.

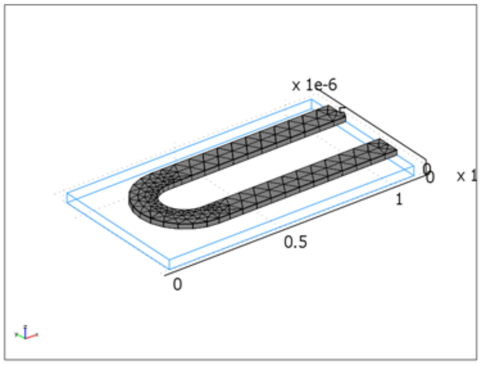
3 In the **Distance** edit field, type `2e-7`.

4 From the **Extrude to geometry** list, select **New geometry**.

5 Click **OK** to extrude the triangular mesh 0.2 µm in the positive *z* direction. This creates a prism mesh in a new 3D geometry, **Geom4 (3D)**.

6 In the **View Geometries** dialog box, select **Geom3 (3D)**. Click **OK** to view both 3D geometries.

7 Click the **Zoom Extents** button on the Main toolbar.



The extruded prism mesh. The blue lines indicate the base block geometry.

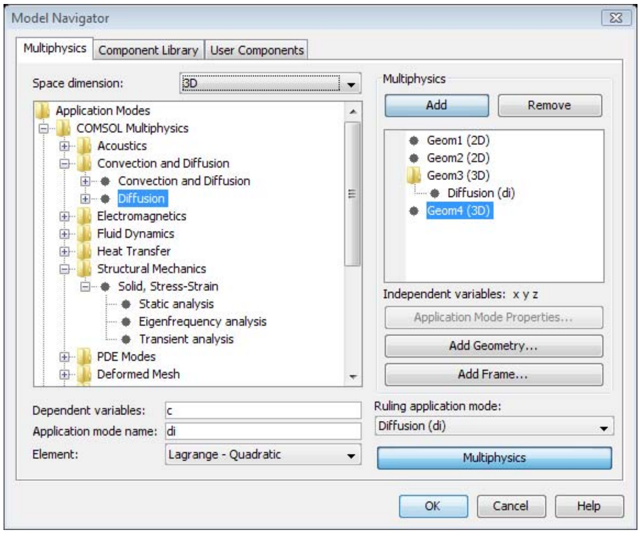
Model Navigator

Open the Model Navigator from the **Multiphysics** menu to add Diffusion application modes to both 3D geometries:

1. From the **Multiphysics** menu, choose the **Model Navigator**.
2. Select **Geom3 (3D)** in the list of geometries in the **Multiphysics** area.
3. In the list of application modes, open the **COMSOL Multiphysics> Convection and Diffusion** folder. Select **Diffusion**, then click **Add**.
4. Select **Geom4 (3D)** in the list of geometries in the **Multiphysics** area.

   Add another Diffusion application mode with the same name and dependent variable:
5. Make sure that **Diffusion** is selected in the list of application modes.
6. Type `c` in the **Dependent variables** edit field.

7 In the **Application mode name** edit field, type `di`.



Space dimension: 3D

Independent variables: x y z

Dependent variables: c

Application mode name: di

Element: Lagrange - Quadratic

Ruling application mode: Diffusion (di)

8 Click **Add**, then click **OK**.

OPTIONS AND SETTINGS

1 From the **Options** menu, choose **Constants**.

2 Enter the constants from the following table; when finished, click **OK**.

TABLE 13-2: MODEL DATA

NAME	EXPRESSION	DESCRIPTION
`r_surf`	`0.005[mol/(m^2*s)`	`Reaction rate, outlet`
`c0`	`7[mol/m^3]`	`Concentration, inlet`
`D1`	`5e-5[m^2/s]`	`Diffusion coefficient, top layer`
`D1_eff`	`1e-6[m^2/s]`	`Effective diffusion coefficient, base block`

PHYSICS SETTINGS

Subdomain Settings—Base Block

1 Click the **Geom3** tab.

2 In the **View Geometries** dialog box, select **Geom4 (3D)**. Click **OK** to view both 3D geometries.

3 From the **Physics** menu, choose **Subdomain Settings**.

4 Select Subdomain 1.

5 Type `D1_eff` in the **D (isotropic)** edit field for the isotropic diffusion coefficient.

6 Click the **Init** tab.

7 In the **c(t_0)** edit field, type `c0`.

8 Click **OK**.

Boundary Conditions—Base Block

1 From the **Physics** menu, choose **Boundary Settings**.

2 Select Boundary 3 (the bottom boundary).

3 From the **Boundary condition** list, select **Flux**.

4 Type `-r_surf*c/c0` in the **N_0** edit field for the inward flux to define an outward flux.

5 Click **OK**.

Subdomain Settings—Top Layer

1 Click the **Geom4** tab.

2 In the **View Geometries** dialog box, select **Geom3 (3D)**. Click **OK** to view both 3D geometries.

3 From the **Physics** menu, choose **Subdomain Settings**.

4 Press Ctrl+A to select all subdomains.

5 Type `D1` in the **D (isotropic)** edit field for the isotropic diffusion coefficient.

6 Click the **Init** tab.

7 In the **c(t_0)** edit field, type `c0`.

8 Click **OK**.

Boundary Conditions—Top Layer

1 From the **Physics** menu, choose **Boundary Settings**.

2 Select Boundary 18 (the top inflow boundary).

3 Select **Concentration** in the **Boundary condition** list.

4 In the **c_0** edit field for concentration, type `c0`.

5 Click **OK**.

Identity Boundary Conditions

To couple the concentration values from the base block to the top layer, use an identity boundary condition.

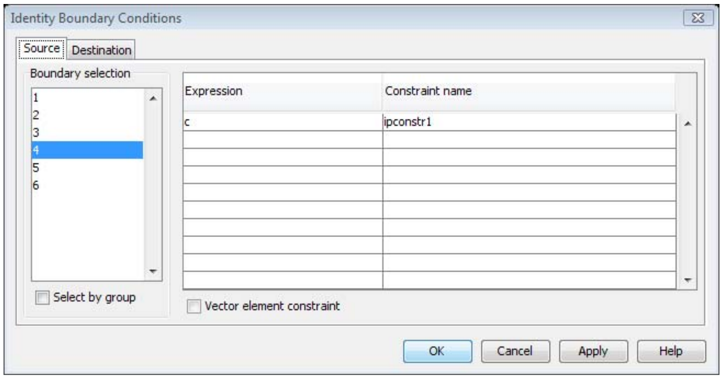
1 Click the **Geom3** tab.

2 From the **Physics** menu, point to **Identity Conditions**, and then click **Identity Boundary Conditions**.

3 Select Boundary 4 (the top boundary).

4 Type `c` in the top row under **Expression**.

5 Move to the **Constraint** name column. The default name **ipconstr1** appears automatically.


Identity Boundary Conditions

Source | Destination

Boundary selection: 1, 2, 3, 4, 5, 6

Select by group

Expression	Constraint name
c	ipconstr1

Vector element constraint

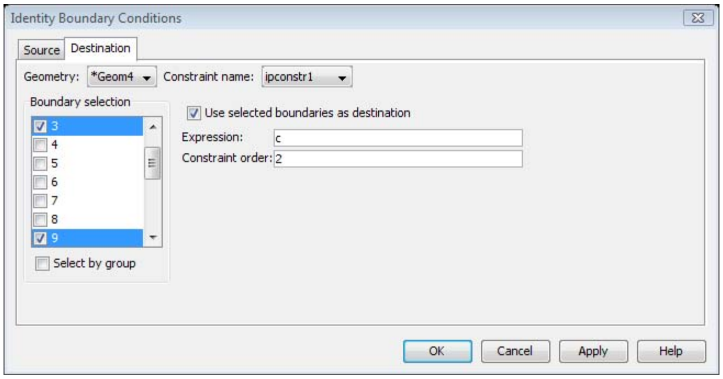
OK Cancel Apply Help

6 Click the **Destination** tab.

7 From the **Geometry** list, select **Geom4**.

8 Select Boundaries 3, 9, and 14 in the user interface. Click to select the **Use selected boundaries as destination** check box or click to select these boundaries directly in the **Boundary selection** list.

9 In the **Expression** edit field, type `c`.


Identity Boundary Conditions

Source | Destination

Geometry: *Geom4 Constraint name: ipconstr1

Boundary selection: ☑ 3, ☐ 4, ☐ 5, ☐ 6, ☐ 7, ☐ 8, ☑ 9

☐ Select by group

☑ Use selected boundaries as destination

Expression: c

Constraint order: 2

OK Cancel Apply Help

10 Click **OK**.

Computing the Solution

Click the **Solve** button on the Main toolbar to start the analysis.

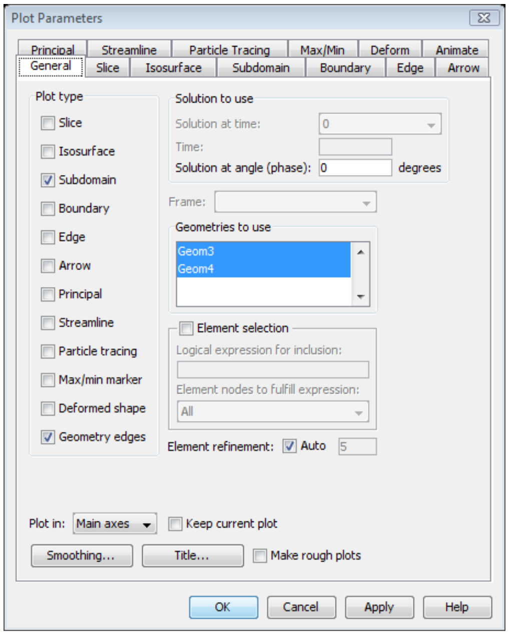
Postprocessing and Visualization

To show the concentration distribution in the entire geometry, do the following steps:

1 From the **Postprocessing** menu, choose **Plot Parameters**.

2 On the **General** page, select both **Geom3** and **Geom4** in the **Geometries to use** list.

3 In the **Plot type** area click to clear the **Slice** check box, then select the **Subdomain** check box.


Plot Parameters

Principal | Streamline | Particle Tracing | Max/Min | Deform | Animate
General | Slice | Isosurface | Subdomain | Boundary | Edge | Arrow

Plot type
- [ ] Slice
- [ ] Isosurface
- [✓] Subdomain
- [ ] Boundary
- [ ] Edge
- [ ] Arrow
- [ ] Principal
- [ ] Streamline
- [ ] Particle tracing
- [ ] Max/min marker
- [ ] Deformed shape
- [✓] Geometry edges

Solution to use
Solution at time: 0
Time:
Solution at angle (phase): 0 degrees

Frame:

Geometries to use
Geom3
Geom4

[ ] Element selection
Logical expression for inclusion:
Element nodes to fulfill expression: All

Element refinement: [✓] Auto 5

Plot in: Main axes
[ ] Keep current plot
[ ] Make rough plots

4 Click **OK**.

14

Deformed Meshes

This chapter explains how to use the application modes that control mesh deformation: the Moving Mesh (ALE) application mode and the Parameterized Geometry application mode.

Deformed Mesh Fundamentals

A deformed mesh can be useful if the boundaries of your computational domain are moving in time or as a function of a parameter. The point is that a new mesh need not be generated for each configuration of the boundaries—instead, the software simply perturbs the mesh nodes so they conform with the moved boundaries.

In COMSOL Multiphysics, you can control the movement of the interior nodes in three ways:

- By propagating the moving boundary displacement throughout the domain to obtain a smooth mesh deformation everywhere — this is done by solving PDEs for the mesh displacements (a Laplace or Winslow smoothing PDE) with boundary conditions given by the movement of the boundaries.
- By specifying an explicit formula for the mesh deformation.
- By letting the mesh movement be determined by some physical deformation variables, such as the displacement components of structural mechanics.

The technique for mesh movement is called an *arbitrary Lagrangian-Eulerian* (ALE) method. In the special case of a *Lagrangian* method, the mesh movement follows the movement of the physical material. Such a method is often used in solid mechanics, where the displacements often are relatively small.

When the material motion is more complicated, like in a fluid flow model, the Lagrangian method is not appropriate. For such models, an *Eulerian* method, where the mesh is fixed, is often used—except that this method cannot account for moving boundaries.

The ALE method is an intermediate between the Lagrangian and the Eulerian method, and it combines the best features of both—it allows moving boundaries without the need for the mesh movement to follow the material.

Mathematical Description of the Mesh Movement

Though moving meshes are also possible in 3D, consider a 2D geometry for simplicity. First denote the spatial coordinates as (x, y) and then let (X, Y) be the coordinates of a mesh node in the initial, undeformed configuration. To describe the spatial coordinates (x, y) of the same mesh node in the deformed configuration, you can use the functions

$$x = x(X, Y, t), \qquad y = y(X, Y, t)$$

where t is time or some other parameter. This coordinate transformation relates two *frames* (coordinate systems):

- The *spatial frame* is the usual, fixed coordinate system with the *spatial coordinates* (x,y). In this coordinate system the mesh is moving, that is, the coordinates (x,y) of a mesh node are functions of time.
- The *reference frame* is the coordinate system defined by the *reference coordinates* (X,Y). In this coordinate system the mesh is fixed to its initial position. You can view the reference frame as a curvilinear coordinate system that follows the mesh.

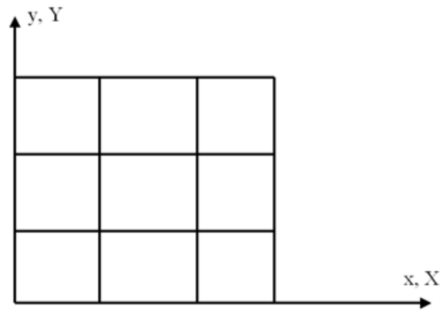


Figure 14-1: An undeformed mesh. The spatial frame (x,y) and the reference frame (X,Y) coincide.

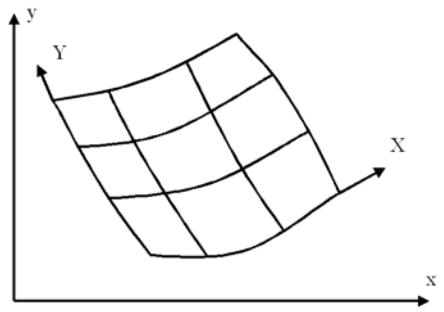


Figure 14-2: A deformed mesh with spatial frame (x,y) and reference frame (X,Y).

Some geometric variables that the software defines are available for both the spatial and the reference frames (see “Geometric Variables” on page 165 in the *COMSOL Multiphysics User’s Guide*).

Derivatives

SPATIAL DERIVATIVES

For a dependent variable u, there are two kinds of spatial derivatives:

- u_x and u_y, the derivatives in the spatial frame. That is, u is considered as a function of x and y (and t). These are typically denoted `ux` and `uy` in the software.
- u_X and u_Y, the derivatives in the reference frame. That is, u is considered as a function of X and Y (and t). These are typically denoted `uX` and `uY` in the software.

In COMSOL Multiphysics, it is only possible to use one of these types of derivatives depending on the frame associated with the application mode that defines u.

TIME DERIVATIVES

There are two kinds of time derivatives related to the two frames:

- The common *spatial time derivative* is related to the spatial frame:

  $$u_t = \left.\frac{\partial u}{\partial t}\right|_{x,y}$$

  The spatial coordinates (x, y) are fixed when computing this derivative; that is, you consider a fixed point in space. This derivative is often denoted `ut` in the software (see page 399).
- The *reference time derivative* (or *mesh time derivative*) is related to the reference frame:

  $$u_{\mathrm{TIME}} = \left.\frac{\partial u}{\partial t}\right|_{X,Y}$$

  The reference coordinates (X,Y) are fixed when computing this derivative; that is, you consider a point that follows the moving mesh, for instance a mesh node. This derivative is denoted `uTIME` in the software.

The two derivatives are related by the chain rule:

$$u_t = u_{\mathrm{TIME}} - u_x x_{\mathrm{TIME}} - u_y y_{\mathrm{TIME}}$$

where (x_{TIME},y_{TIME}) is the mesh velocity. The reference time derivative is often less important from the user point view because its value depends on the mesh movement, which has no physical significance. However, for the special case when the mesh follows the material's motion (the Lagrangian method), the reference time derivative is physically significant and is also called the *material time derivative*.

The solvers assemble the discretized model on the deformed mesh, using the reference time derivative of the solution as input. They also compute the spatial time derivative by the preceding formula. This implies that the PDEs do not have to be reformulated when you use a deformed mesh. However, if you use a Lagrangian method, some material properties might depend on the material deformation (which is then the same as the mesh deformation).

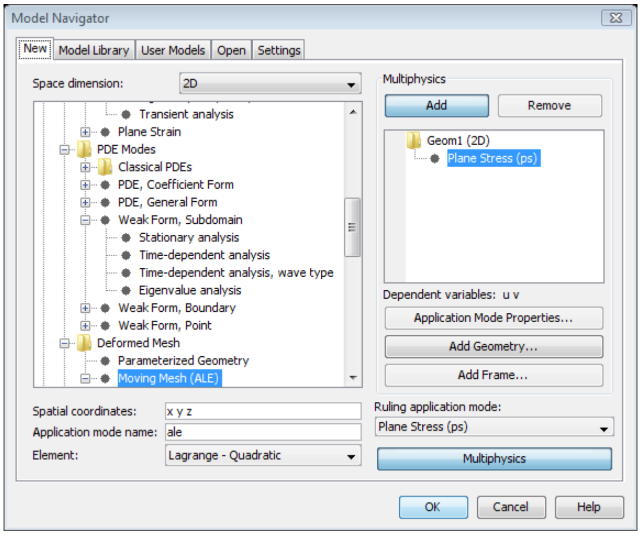
Frames for Deformed Meshes

When modeling deformed meshes you need two *frames*. A *frame* is a coordinate system, and it includes a set of coordinate names. The two frames are:

- The *reference frame*, which is the frame in which the mesh is fixed. This is the frame in which you draw the geometry and gives the shape of the geometry before it is deformed. This frame is associated with the geometry, and it exists even when you do not use deformed meshes.
- The *spatial frame*, which is the usual coordinate system that is fixed in space. In this frame the mesh is moving according to the settings in the deformed mesh application mode. The spatial frame is associated with the deformed mesh application mode.

Frames in the Model Navigator

Before you have added a deformed mesh application mode, there is only one frame, the reference frame. The reference coordinates are the independent variables in the geometry, x, y, z, by default. You can change these names and the name of the reference frame in the **Add Geometry** dialog box.



Space dimension: 2D

Dependent variables: u v

Spatial coordinates: x y z

Application mode name: ale

Element: Lagrange - Quadratic

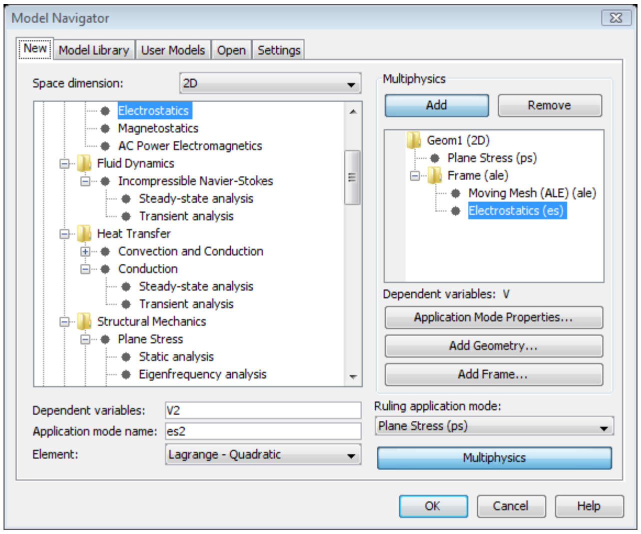
Ruling application mode: Plane Stress (ps)

Figure 14-3: The Model Navigator with a deformed mesh application mode selected in the list to the left.

When you select a deformed mesh application mode from the list of application modes to the left in the Model Navigator, you can specify the names of the spatial coordinates in an edit field below the list. By default they take the names of the independent variables of the geometry. If you do not change the default, the independent variables of the geometry (the reference coordinates) get new names. The software creates the names by capitalizing the spatial coordinates. For example, if the spatial coordinates are x, y, and z, it renames the reference coordinates to X, Y, and Z. If you edit the default spatial coordinates so that they are not equal to the current reference coordinates, the reference coordinates retain their names.

When you add a deformed mesh application mode, the software automatically adds a spatial frame. Its name coincides with the name of the deformed mesh application mode with which it is associated (either ale or pg). Alternatively, you can add the spatial frame in advance by going to the **Multiphysics** area on the right side of the Model Navigator and clicking the **Add Frame** button. This opens the **Add Frame** dialog box, where you can specify the names of the spatial coordinates and the spatial frame. If you then add a deformed mesh application mode, it is associated with the previously defined frame.

The **Multiphysics** area to the right displays the spatial frame as a node attached to the geometry node. The deformed mesh application mode—in this case a Moving Mesh (ALE) application mode—is attached to the spatial frame it defines. All other application modes are attached to the frame in which their equations are formulated. Application modes attached directly to the geometry are associated with the reference frame.



Dependent variables: V

Dependent variables: V2

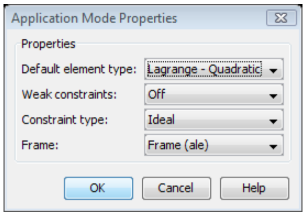
Application mode name: es2

Element: Lagrange - Quadratic

Ruling application mode: Plane Stress (ps)

Figure 14-4: The Model Navigator with a Moving Mesh (ALE) application mode in the model. The Plane Stress application mode's equations are formulated on the reference frame and the Electrostatics application mode's equations are formulated on the frame defined by the Moving Mesh (ALE) application mode.

When you add an application mode, it becomes attached to the reference frame or the spatial frame depending on which node you selected in the **Multiphysics** area. If you want to change the frame to which an application mode is attached, open the Application Mode Properties dialog box and select a frame in the **Frame** list



Default element type: Lagrange - Quadratic
Weak constraints: Off
Constraint type: Ideal
Frame: Frame (ale)

Figure 14-5: The Application Mode Properties dialog box. The Frame property indicates which frame the application mode uses.

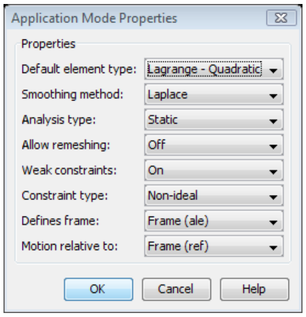
In normal situations, follow these guidelines when attaching an application mode to a frame:

- When using the Parameterized Geometry application mode, attach all application modes to the spatial frame.
- When using the Moving Mesh (ALE) application mode, attach structural mechanics application modes to the reference frame. This is because a structural analysis computes the mesh displacements that you typically use as input for the Moving Mesh (ALE) application mode when it defines the spatial frame. Attach all other application modes to the spatial frame defined by the Moving Mesh (ALE) application mode. In, for example, fluid-structure interaction models, consider taking the area effect into account, because the fluid force is defined on the deformed mesh in the spatial frame whereas the structural mechanics loads use the fixed reference frame. You specify this factor as the mesh element scale factor for the spatial frame divided by the mesh element scale factor for the reference frame. This expression is typically `dvol_ale/dvol`. You find the names of the mesh element scale factors in the **Frames** area of the **Model Settings** dialog box on the **Physics** menu.

An application mode's frame affects the names of its variables. Specifically, the software uses the frame's coordinate names when forming derivatives and vector components. In addition, it interprets a time derivative `ut` as a derivative in the application mode's frame, while `uTIME` is always a derivative in the reference frame. See also "Shape Function Variables" on page 171 in the *COMSOL Multiphysics User's Guide*.

The two application modes defining the deformed mesh have two frame properties:

- The **Defines frame** property indicates which frame's spatial coordinates that the application mode defines (the spatial frame). This frame cannot be the reference frame associated with the geometry.
- The **Motion relative to** property indicates which frame serves as the reference frame. By default this is the reference frame associated with the geometry. It cannot be the same frame as the frame that the application mode defines.


Application Mode Properties

Properties

Default element type:	Lagrange - Quadratic
Smoothing method:	Laplace
Analysis type:	Static
Allow remeshing:	Off
Weak constraints:	On
Constraint type:	Non-ideal
Defines frame:	Frame (ale)
Motion relative to:	Frame (ref)

Figure 14-6: The Application Mode Properties dialog box of the Moving Mesh (ALE) application mode.

Settings for Frames

In the **Model Settings** dialog box, which you open from the **Physics** menu, you can control some settings for each frame (see "Model Settings with Multiple Frames" on page 181 of the *COMSOL Multiphysics User's Guide*). In particular, you can select the order of the shape functions that describe the moving mesh. This order must agree with the element order you choose in the **Subdomain Settings** dialog box for the deformed mesh application mode (see "Subdomain Settings" on page 401). The default choice **Automatic** takes care of this.

Frame Selection for Coupling Variables

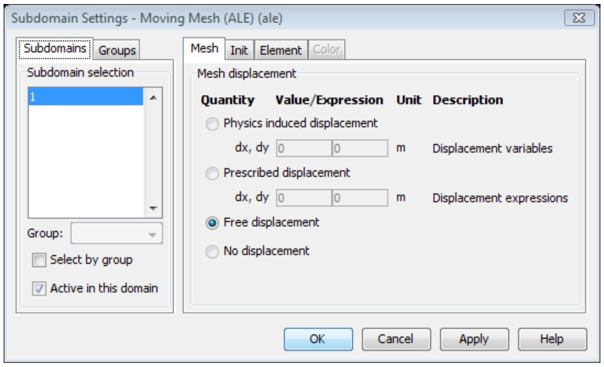
If you work with coupling variables, periodic conditions, or identity conditions when you have multiple frames, you need to select the appropriate frame in the corresponding dialog boxes. See "Integrating on a Deformed Mesh" on page 257 of the *COMSOL Multiphysics User's Guide* for details.

The Moving Mesh Application Mode

With the Moving Mesh (ALE) application mode you can create models where the geometry (actually the mesh) changes shape due to some physics in the model. You can study both static and time-dependent deformations where the geometry changes its shape due to the dynamics of the problem.

Subdomain Settings

The **Subdomain Settings** dialog box in this application mode contains properties pertaining to the moving mesh.



Subdomain Settings - Moving Mesh (ALE) (ale)

Subdomains | Groups

Subdomain selection: 1

Group:

☐ Select by group

☑ Active in this domain

Mesh | Init | Element | Color

Mesh displacement

Quantity	Value/Expression	Unit	Description
○ Physics induced displacement			
dx, dy	0 0	m	Displacement variables
○ Prescribed displacement			
dx, dy	0 0	m	Displacement expressions
● Free displacement			
○ No displacement			

Figure 14-7: The Subdomain Settings dialog box of the Moving Mesh application mode.

On the **Mesh** tab you can specify how the software computes the mesh displacement in each subdomain. The available options are:

- **Free displacement**. This is the default setting, and it means that the mesh displacement is constrained only by the boundary conditions on the surrounding boundaries. The displacement in the subdomain is obtained by solving a PDE (see “Smoothing Methods” on page 404).
- **Physics induced displacement**. Use this option when another application mode calculates the displacement. Normally that other application mode is a structural mechanics application mode, and the displacement variables to enter in the edit fields are the dependent variables of the structural mechanics application mode. In other words, a Lagrangian method is used where the mesh movement follows the material motion. The contents of the edit fields for the displacement variables must be a single dependent variable discretized with Lagrange shape functions. The order

of these shape functions must be compatible with the order chosen on the **Element** tab.

- **Prescribed displacement**. Use this when expressions define the displacement. These should not depend on variables that the model solves. If they do, this dependence does not contribute to the Jacobian matrix, which can cause convergence difficulties. This is useful, for example, in axisymmetric structural models, where the dependent variable represents u/r rather than the displacement u itself, which you have access to as an expression variable.
- **No displacement**. The mesh is not deformed at all.

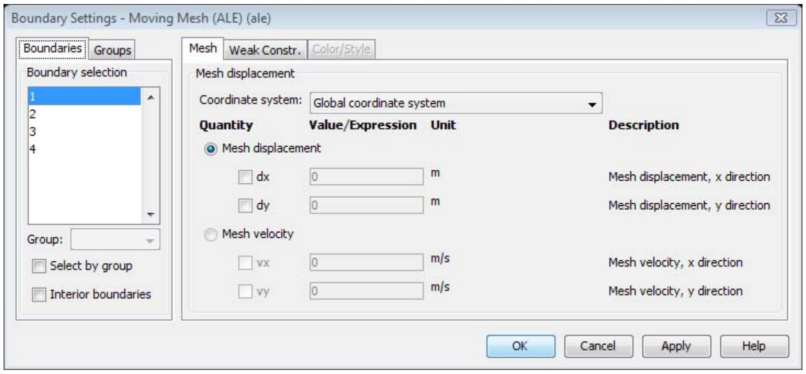
The physics-induced displacement and prescribed displacement might seem very similar because they both require that you enter the displacements as expressions. The difference is that you can use the physics-induced displacement only when the displacement is equal to the dependent variables of an application mode. You cannot use it for general expressions. The prescribed displacement, on the other hand, assumes that the expressions are functions of variables whose values are known (the reference coordinates, for instance).

On the **Init** tab you can give initial values for the mesh positions `x`, `y`, and `z` (if the Laplace smoothing method is used), or the mesh displacements `dx`, `dy`, and `dz` (if the Winslow smoothing method is used).

On the **Element** tab you can select the order of the Lagrange shape functions that define the mesh positions `x`, `y`, and `z` (if the Laplace smoothing method is used), or the mesh displacements `dx`, `dy`, and `dz` (if the Winslow smoothing method is used). The shape functions are only used on subdomains where **Free displacement** has been selected. You can also select the **Integration order** and **Constraint order** used when discretizing the PDE for the mesh movement and its boundary conditions.

Boundary Conditions

At the boundaries you can set constraints on the mesh displacement or, in the transient case, on the mesh velocity. You can set constraints on the exterior boundaries of the subdomains where you use free displacement.


Boundary Settings - Moving Mesh (ALE) (ale)

Boundary selection: 1, 2, 3, 4

Group:

☐ Select by group

☐ Interior boundaries

Mesh displacement

Coordinate system: Global coordinate system

Quantity	Value/Expression	Unit	Description
● Mesh displacement			
☐ dx	0	m	Mesh displacement, x direction
☐ dy	0	m	Mesh displacement, y direction
○ Mesh velocity			
☐ vx	0	m/s	Mesh velocity, x direction
☐ vy	0	m/s	Mesh velocity, y direction

Figure 14-8: Boundary Settings dialog box of the Moving Mesh application mode.

If you want to constrain a certain displacement component, select its check box and enter a expression in the **Mesh displacement** edit field. If the check box is not selected, the boundary gets a Neumann boundary condition. You can also select in which coordinate system the displacement is specified.

For example, if you have a so-called free boundary, it is appropriate to constrain the normal component of the mesh velocity to be equal to the normal component of the material velocity. You can do this by selecting the **Tangent and normal** coordinate system, entering the expression `nx*u+ny*v` for the normal velocity, and using a Neumann condition on the tangential velocity. This setup allows the mesh to slip relative to the material along the boundary, which improves mesh quality. Note that (`nx`, `ny`) denotes the normal of the deformed mesh, while (`nX`, `nY`) denotes the normal of the undeformed mesh. Similar naming conventions hold for other geometric variables, see "Geometric Variables" on page 165 in the *COMSOL Multiphysics User's Guide*.

If the boundary condition involves variables from another application mode (like in the free boundary condition), the usual pointwise constraints or ideal weak constraints make unwanted modifications of the boundary condition for the other application mode. For this reason, the Moving Mesh (ALE) application mode uses non-ideal weak constraints by default.

It is important that the boundary conditions are consistent with the settings in the adjacent subdomains. If the subdomain settings define a certain displacement, the boundary must be displaced in the same way.

Smoothing Methods

In the domains with free displacement, the Moving Mesh (ALE) application mode solves an equation for the mesh displacement. This equation smoothly deforms the mesh given the constraints you place on the boundaries. You can choose between *Laplace smoothing* and *Winslow smoothing*. To specify the smoothing methods, use the **Application Mode Properties** dialog box, which you access from the **Multiphysics** area in the Model Navigator or from the **Physics** menu.

To see how these smoothing methods differ, let x and y be the spatial coordinates of the frame which the application mode defines, and let X and Y be the reference coordinates of the reference frame. If you choose Laplace smoothing, the software introduces deformed mesh positions x and y as degrees of freedom in the model. In the static case, it solves the equation

$$\frac{\partial^2 x}{\partial X^2}+\frac{\partial^2 x}{\partial Y^2} = 0$$

and in the transient case it solves the equation

$$\frac{\partial^2}{\partial X^2}\frac{\partial x}{\partial t}+\frac{\partial^2}{\partial Y^2}\frac{\partial x}{\partial t} = 0$$

Similar equations hold for the y coordinate.

If you choose Winslow smoothing, the software solves the equation

$$\frac{\partial^2 X}{\partial x^2}+\frac{\partial^2 X}{\partial y^2} = 0$$

and does the same for Y. Equivalently, X and Y satisfy Laplace equations as functions of the x and y coordinates.

COMPATIBILITY WITH 3.2 MODELS

COMSOL Multiphysics 3.2 used a different implementation of the Winslow smoothing. The old implementation used mesh displacements instead of mesh positions as degrees of freedom. For backward compatibility, 3.2 models that use

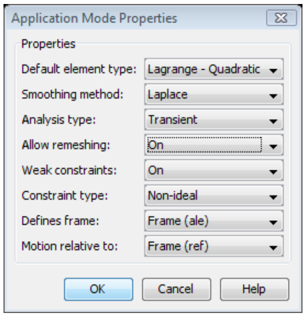
Winslow smoothing retain the old implementation. You can change the choice of smoothing method in the **Application Mode Properties** dialog box. The old implementation of the Winslow smoothing does not support remeshing.

Remeshing

If the mesh displacement becomes large, the mesh elements eventually have a very bad quality or even become inverted. Depending on the specific model, either smoothing technique may do a better job avoiding inverted mesh elements. Once a mesh element becomes inverted, it is no longer possible to solve any other equation on that frame. This section describes how to remesh the geometry when this happens and how to continue solving.

ENABLING REMESHING

You enable the remeshing support in the Moving Mesh (ALE) application mode in the Application Mode Properties dialog box. Set the property **Allow remeshing** to **On**. You must make this selection before solving the model. If you attempt to solve it and later realize that you need to remesh while remeshing support is not activate, you must go back, activate it, and re-solve the model.


Application Mode Properties

Properties

Default element type:	Lagrange - Quadratic
Smoothing method:	Laplace
Analysis type:	Transient
Allow remeshing:	On
Weak constraints:	On
Constraint type:	Non-ideal
Defines frame:	Frame (ale)
Motion relative to:	Frame (ref)

FRAMES WHEN USING REMESHING

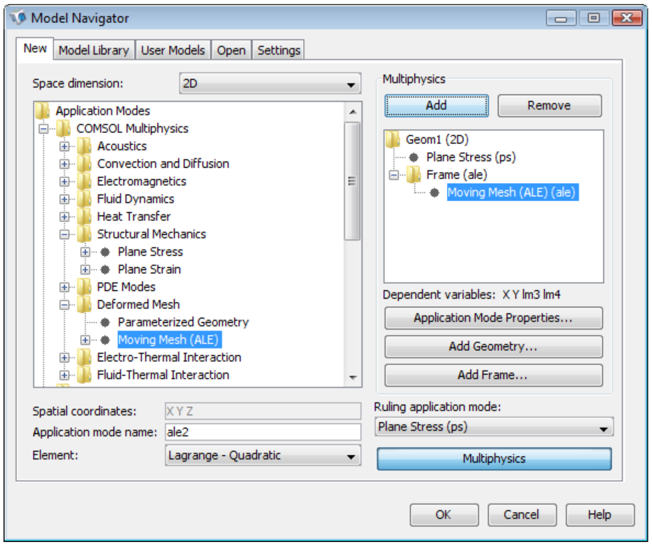
When you set the application mode property **Allow remeshing** to **On**, the software adds an extra frame. The three frames are now:

- The *spatial frame*, usually denoted by `ale` or `pg` (the name of the application mode controlling the deformed mesh). This is the usual coordinate system. The corresponding coordinates are called `x` and `y` by default.
- The *reference frame*, usually denoted by `ref`. This coordinate system describes the original configuration (before remeshing). The corresponding coordinates are

called X and Y by default. If the mesh movement follows the material movement (as is common in solid mechanics), this frame is also called the *material frame*. In this case, X and Y are the coordinates of the material point in the original configuration. The software stores the values of X and Y for all nodes as degrees of freedom in the solution vector but never solves for them. Their values only change when mapping the solution, which happens when you click the **Restart** button after a remeshing.

- The *mesh frame*, usually denoted by `mesh`. This coordinate system describes the configuration just after the latest remeshing. The corresponding coordinates are called `Xm` and `Ym` by default.

The guidelines on page 399 regarding which frame to attach the physics application modes to still apply. Note that no physics application mode should be attached to the mesh frame. In the **Model Navigator** the mesh frame is associated with the geometry and the reference frame is a separate node. This differs from the case without remeshing where the reference frame was associated with the geometry.



Space dimension: 2D

Dependent variables: X Y lm3 lm4

Spatial coordinates: X Y Z

Application mode name: ale2

Element: Lagrange - Quadratic

Ruling application mode: Plane Stress (ps)

STOP CONDITION

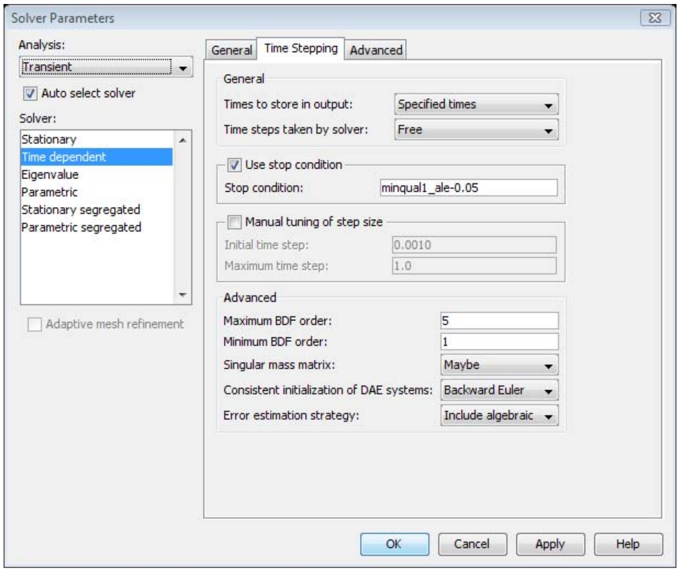
To make the solver stop when the mesh quality is poor you need to activate a stop condition in the **Solver Parameters** dialog box. Such a stop condition exists for the time-dependent solver and the parametric solver.

The solver halts when the expression in the **Stop condition** edit field becomes negative. The default value for the stop condition is `minqual1_ale-0.05`. The Moving Mesh

(ALE) application mode defines the variable `minqual1_ale`, which represents the minimum quality of the deformed mesh. The variable name begins with `minqual` followed by the geometry number. This nomenclature creates unique variables in the case where a model has multiple geometries. For models with one geometry this number is 1, resulting in `minqual1`. The variable name for the minimum quality ends with an underscore (_) followed by the name of the frame for the deformed mesh. In this case, the name of that frame is `ale`, which means that the name of the quality variable becomes `minqual1_ale`. To find the name of the frame, open the **Model Navigator** and see to which frame the Moving Mesh (ALE) application mode is attached. You can also look in the **Application Mode Properties** dialog box. The **Defines frame** property gives the name of the frame of the deformed mesh.

The default stop condition makes the solver should stop when the minimum quality is less than 0.05. Depending on the initial quality of the mesh, you might have to change this number.

If the model has more than one geometry with a deformed mesh, the stop condition uses the minimum quality of all the geometries. In a model with two geometries, use the stop condition `min(minqual1_ale,minqual2_ale)-0.05`.



Analysis: Transient

[✓] Auto select solver

Solver: Time dependent

[ ] Adaptive mesh refinement

General

Times to store in output: Specified times

Time steps taken by solver: Free

[✓] Use stop condition

Stop condition: minqual1_ale-0.05

[ ] Manual tuning of step size

Initial time step: 0.0010

Maximum time step: 1.0

Advanced

Maximum BDF order: 5

Minimum BDF order: 1

Singular mass matrix: Maybe

Consistent initialization of DAE systems: Backward Euler

Error estimation strategy: Include algebraic

REMESHING STEP BY STEP

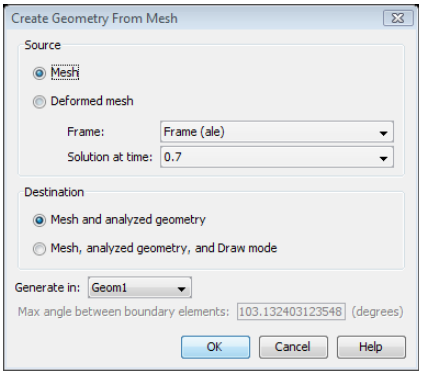
To solve using remeshing of a deformed mesh follow these steps:

1 Activate the remeshing support in the application mode by setting the property **Allow Remeshing** to **On** in the **Application Mode Properties** dialog box.

2 Specify the stop condition in the **Solver Parameters** dialog box.

3 Solve. If the solver stops before is has reached the final time continue with the next step below.

4 Next you need to create a new geometry from the deformed mesh. Open the **Create Geometry From Mesh** dialog box. Click the **Deformed Mesh** radio button to use the deformed mesh as the source for the new geometry. You can choose at which time to pick the mesh but in general use the last one.


Create Geometry From Mesh

Source

[✓ Mesh] Deformed mesh

Frame: Frame (ale)

Solution at time: 0.7

Destination

[✓ Mesh and analyzed geometry] Mesh, analyzed geometry, and Draw mode

Generate in: Geom1

Max angle between boundary elements: 103.132403123548 (degrees)

5 Mesh the new geometry just created. This mesh gets a better quality than the previous deformed mesh.

6 In the **Solver Parameters** dialog box change the initial time (or the first parameter value if you are using the parametric solver) to the last value that the solver reached.

7 Click the **Restart** button to continue solving. This uses the previous solution as the initial value and thus continue solving where the solver had stopped.

8 If the solver again stops before the final time go to Step 4 again.

REMESHING—A SIMPLE EXAMPLE

The section describes the remeshing technique by building a simple model where a force bends a beam.

1 In the **Model Navigator**, select **2D** from the **Space dimension** list.

2 In the **COMSOL Multiphysics** folder, select **Deformed Mesh>Moving Mesh (ALE)>Transient analysis**.

3 Click the **Multiphysics** button and then the **Add** button to add the Moving Mesh application mode to the model.

4 In the list of application modes, select **Structural Mechanics>Plane Stress>Transient analysis**. In the **Multiphysics** area select the node **Geom1 (2D)**. Then click **Add** to add the Plane Stress application mode.

5 Click **OK** to close the **Model Navigator**.

Geometry Modeling

1 Draw a rectangle with a width of 1.6, height of 1.2, and with its lower left corner at (–0.6, –0.4).

2 Draw a second rectangle with a width of 1.4, height of 0.2, and its lower left corner at (–0.6, –0.2).

Physics Settings

1 In the **Subdomain Settings** dialog box of the Plane Stress application mode clear the **Active in this domain** check box for Subdomain 1 (the air). Click **OK** to close the dialog box.

2 In the **Boundary Settings** dialog box select Boundary 3 and activate the constraints R_x and R_y by selecting the corresponding check boxes. Go to the **Load** tab and select Boundaries 4, 6, and 8. Enter a force that increases with time by setting F_y to `-1e6*t`. Click **OK** to close the dialog box.

3 Select the **Moving Mesh (ALE)** application mode from the **Multiphysics** menu.

4 In the **Subdomain Settings** dialog box of the **Moving Mesh** application mode select Subdomain 2 and click the **Physics induced displacement** button. Enter the displacement variables `u` and `v` in the **dx, dy** edit fields. Click **OK** to close the dialog box.

5 In the **Boundary Settings** dialog box select Boundaries 4, 6, and 8. Select the **dx** and **dy** check boxes and then enter the mesh displacement `u` and `v` in the corresponding edit fields.

6 Select Boundaries 1, 2, 5, 7, and 9, and select the **dx** and **dy** check boxes to prevent these boundaries from moving. Click **OK** to close the dialog box.

Note: To obtain a correct solution, you should use the support for large deformations in the Plane Stress application mode, but that feature is available only in the extended Plane Stress application mode in the Structural Mechanics and MEMS modules.

The next step is to activate support for remeshing in the **Moving Mesh** application mode:

1 Open the **Application Mode Properties** dialog box by selecting **Properties** from the **Physics** menu.
2 Select **On** from the **Allow remeshing** list and click **OK**.

You must make this selection before solving the model. If you attempt to solve it and later realize that you need to remesh while remeshing support is not activate, you must go back, activate it, and re-solve the model.

Mesh Generation

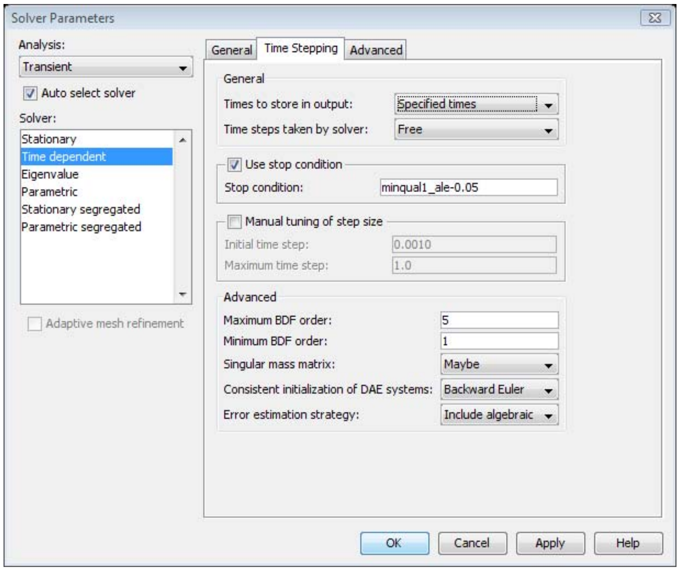
1 Initialize the mesh.
2 Refine it once.

Computing the Solution

To make the solver stop when the mesh quality becomes poor, first enable a stop condition in the **Solver Parameters** dialog box.

1 From the **Solve** menu open the **Solver Parameters** dialog box. In the **Times** edit field enter `0:0.01:0.3`.

2 Go to the **Time Stepping** page and select the **Use stop condition** check box.


Solver Parameters

Analysis: Transient

[✓] Auto select solver

Solver: Stationary, [✓ Time dependent], Eigenvalue, Parametric, Stationary segregated, Parametric segregated

[ ] Adaptive mesh refinement

General

Times to store in output: Specified times

Time steps taken by solver: Free

[✓] Use stop condition

Stop condition: minqual1_ale-0.05

[ ] Manual tuning of step size

Initial time step: 0.0010

Maximum time step: 1.0

Advanced

Maximum BDF order: 5

Minimum BDF order: 1

Singular mass matrix: Maybe

Consistent initialization of DAE systems: Backward Euler

Error estimation strategy: Include algebraic

3 Click **OK** to close the **Solver Parameters** dialog box and then click the **Solve** button on the Main toolbar.

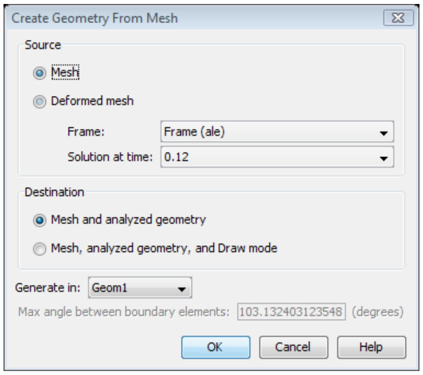
The solver stops somewhere around $t = 0.13$.

Geometry Modeling

Next you need to create a new geometry and mesh from the deformed mesh.

1 From the **Mesh** menu open the **Create Geometry From Mesh** dialog box.

2 Click the **Deformed mesh** button, then click **OK**.


Create Geometry From Mesh

Source

[✓ Mesh]

Deformed mesh

Frame: Frame (ale)

Solution at time: 0.12

Destination

[✓ Mesh and analyzed geometry]

Mesh, analyzed geometry, and Draw mode

Generate in: Geom1

Max angle between boundary elements: 103.132403123548 (degrees)

This creates a new geometry from the deformed mesh at the last time step.

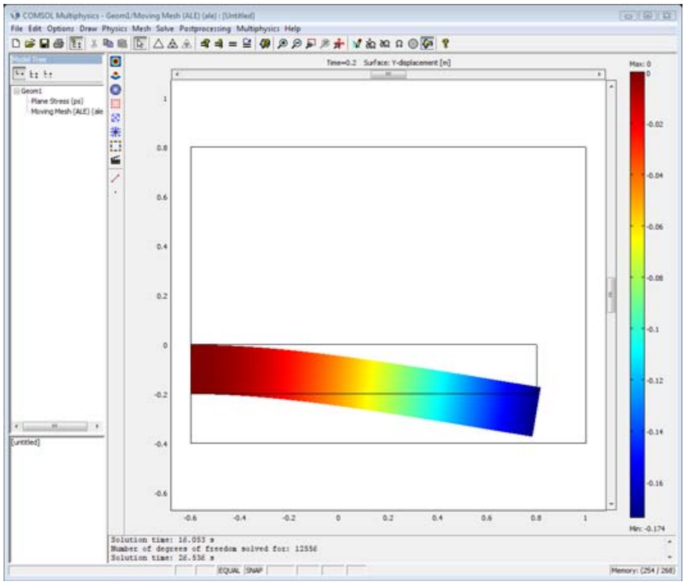
Mesh Generation

Using buttons on the Main toolbar, initialize the mesh and refine it once. This creates a new mesh with higher quality for the deformed geometry.

Computing the Solution

1 Open the **Solve>Solver Parameters** dialog box and on the **General** tab change the list of times in the **Times** edit field to start from the last time of the current solution. Generally when solving time-dependent problems this is not absolutely necessary, but in this case it is important because the force in the Plane Stress application mode explicitly depends on the time.
2 When you continue solving the model, it is important to use the solution from the last time step as the new initial value when restarting the solver. Do this by clicking the **Restart** button on the Main toolbar.

The solver continues solving and stops when the mesh quality again becomes poor.


COMSOL Multiphysics - Geom1/Moving Mesh (ALE) (ale) : [Untitled]
File Edit Options Draw Physics Mesh Solve Postprocessing Multiphysics Help
Geom1
Plane Stress (ps)
Moving Mesh (ALE) (ale
Time=0.2 Surface: Y-displacement [m]
Max: 0
Min: -0.174
EQUAL SNAP
Memory: (254 / 268)

```
Solution time: 18.053 s
Number of degrees of freedom solved for: 12556
Solution time: 28.538 s
```

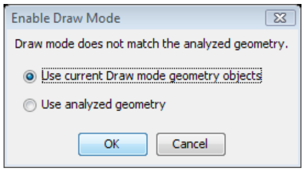
POSTPROCESSING

To postprocess the two parts of the simulation together, you can export the FEM structure to the command line after each part of the simulation. Use

File>Export>Export FEM Structure to save each FEM structure with a new name. Then use the ability of the `postmovie` command to postprocess several FEM structures.

RESTORING THE ORIGINAL GEOMETRY

Assume you want to go back and restart the solver from the very beginning. In this case you first must restore the original geometry. To do so, go to the Draw mode. Because you have generated a deformed geometry, the objects in Draw mode do not represent the current geometry. The software therefore asks if you want to use the current objects in Draw mode or replace them with the deformed geometry (see "Entering Draw Mode" on page 36 in the *COMSOL Multiphysics User's Guide*). Using the current Draw mode objects restores the original geometry.


Enable Draw Mode

Draw mode does not match the analyzed geometry.

[✓ Use current Draw mode geometry objects]

Use analyzed geometry

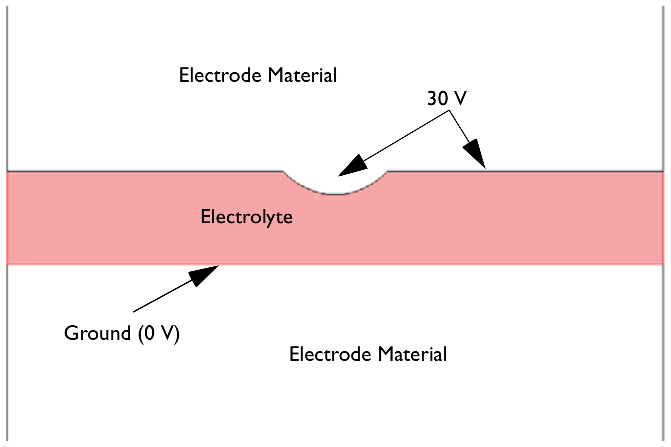
Example: Electrochemical Polishing

Introduction

This example illustrates the principle of electrochemical polishing. The simplified 2D model geometry consists of two electrodes and an intermediate electrolyte domain The positive electrode has a protrusion, representing a surface defect. The purpose of the model is to examine how this protrusion and the surrounding electrode material are depleted over a period of time.

Model Definition

The potential drop over the electrodes is 30 V, and the electrolyte has a conductivity of 10 S/m.



Modeling the depletion of the positive electrode requires a moving boundary because the geometry changes and the current density distribution with it. A simple model for the depletion is based on the assumption that the depletion rate is proportional to the normal current density at the electrode surface. The velocity, U, normal to the mesh at the electrode surface then becomes

$$U = -KJ_n$$

where K is the coefficient of proportionality, and J_n is the normal current density. In this model, $K = 10^{-11}\ \mathrm{m^3/As}$.

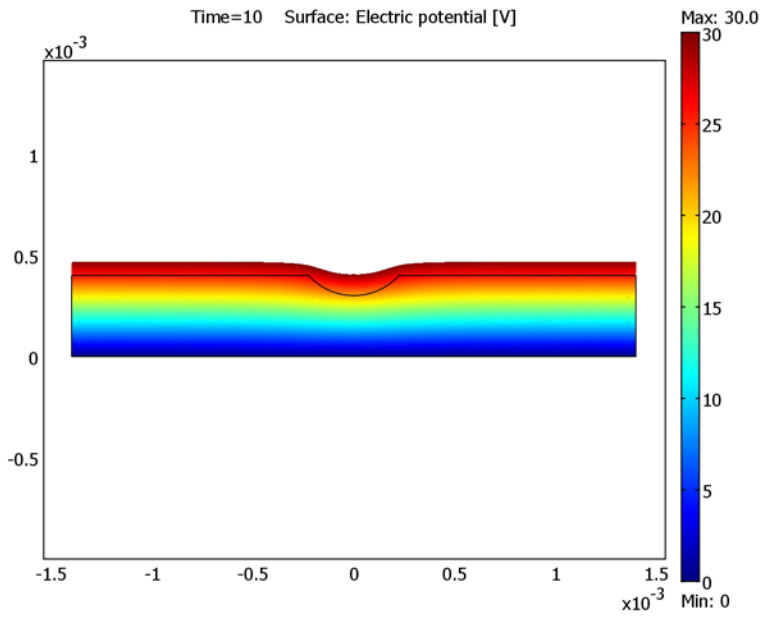
The part of the electrode and electrolyte that the model includes is about 3 mm wide and the distance between the electrodes is 0.4 mm.

Modeling in COMSOL Multiphysics

This model uses the Conductive Media DC and transient Moving Mesh (ALE) application modes. The variable for the normal current density defines the mesh velocity. The dynamics of the model is quasi-static in nature, and the time-dependency only enters in the depletion (removal of material) of the electrode.

Results

After 10 s, the protrusion is somewhat smoothed out, and a significant portion of the positive electrode has been depleted:



Model Library path: `COMSOL_Multiphysics/Electromagnetics/electrochemical_polishing`

MODEL NAVIGATOR

1 Select **2D** in the **Space dimension** list. (This model can easily be generalized to 3D.)

2 In the application mode list, open **COMSOL Multiphysics**>**Deformed Mesh** and then **Moving Mesh (ALE)>Transient analysis**.

3 Click the **Multiphysics** button and then the **Add** button.

4 In the application mode list, open **COMSOL Multiphysics> Electromagnetics>Conductive Media DC**.

5 Click the **Add** button. This adds the **Conductive Media DC** application mode to the moving mesh frame.

6 Click **OK** to close the **Model Navigator**.

OPTIONS

1 From the **Options** menu, open the **Constants** dialog box.

2 Enter a constant with the name `K`, the expression `1e-11[m^3/(A*s)]`, and the description `Coefficient of proportionality` (the description is optional).

3 Click **OK** to close the **Constants** dialog box.

GEOMETRY MODELING

1 Click the **Rectangle/Square** button on the Draw toolbar, and draw a rectangle with corners at $x = -1.4$, $y = 0$ and $x = +1.4$, $y = 0.4$.

2 Shift-click the **Ellipse/Circle (Centered)** button.

3 In the Circle dialog box, type `0.3` in the **Radius** edit field and set the center coordinates to (0, 0.6) by typing `0` in the **X** edit field and `0.6` in the **Y** edit field.

4 Click **OK**.

5 Select both objects by using the keyboard shortcut Ctrl+A. Both objects are now highlighted in red.

6 Click the **Difference** button on the Draw toolbar to remove the circle object from the rectangle. This creates the protruding part of the electrode.

7 The model needs to be the order of mm and not m. Click to select the object and then click the **Scale** button on the Draw toolbar.

8 Enter a scale factor of 10^{-3} by typing `1e-3` in both the **X** and **Y** edit fields in the **Scale factor** area. Click **OK**.

9 Click the **Zoom Extents** button on the Main toolbar.

The geometry is now ready.

PHYSICS SETTINGS

Subdomain Settings—Conductive Media DC

1 From the **Multiphysics** menu, select the **Conductive Media DC** application mode.

2 From the **Physics** menu, select **Subdomain Settings**.

3 In the **Subdomain Settings** dialog box, select Subdomain 1 (the only one) and type `10` in the **Electric conductivity** edit field.

4 Click **OK** to close the **Subdomain Settings** dialog box.

Boundary Settings—Conductive Media DC

1 From the **Physics** menu, select **Boundary Settings**.

2 For Boundaries 3, 4, 6, and 7, select the **Electric potential** in the **Boundary condition** list and type `30` in the **Electric potential** edit field.

3 For Boundaries 1 and 5, select **Electric insulation** in the **Boundary condition** list. Electric insulation is a good approximation if you want to simulate that the electrodes are extended indefinitely in both directions.

4 For Boundary 2, select **Ground** as the boundary condition.

5 Click **OK** to close the **Boundary Settings** dialog box.

Boundary Settings—Moving Mesh (ALE)

1 From the **Multiphysics** menu, select the **Moving Mesh (ALE)** application mode.

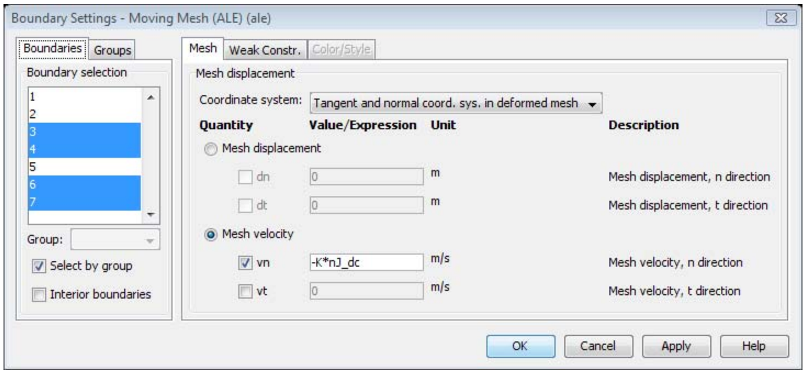
2 From the **Physics** menu, select **Boundary Settings**.

3 In the **Boundary Settings** dialog box, for boundary 2, click the **Mesh displacement** button and set the mesh displacement in both directions to `0` by selecting the **dx** and **dy** check boxes.

4 For Boundaries 1 and 5, click the **Mesh velocity** button and set the mesh velocity, in the x direction to 0 by selecting the **vx** check box.

5 Select Boundaries 3, 4, 6, and 7, and select **Tangent and normal coord. sys. in deformed mesh** in the **Coordinate system** list. Click the **Mesh velocity** button. Then select the **vn** check box and type `-K*nJ_dc` in the **Mesh velocity, n direction** edit field. This makes the normal mesh velocity equal to $-KJ_n$ (`nJ_dc` is the variable for the current density outflow from the Conductive Media DC application mode).

6 Click **OK**.


Boundary Settings - Moving Mesh (ALE) (ale)

Boundary selection: 1, 2, 3, 4, 5, 6, 7 (3, 4, 6, 7 selected)

Group:

☑ Select by group

☐ Interior boundaries

Mesh displacement

Coordinate system: Tangent and normal coord. sys. in deformed mesh

Quantity	Value/Expression	Unit	Description
○ Mesh displacement			
☐ dn	0	m	Mesh displacement, n direction
☐ dt	0	m	Mesh displacement, t direction
● Mesh velocity			
☑ vn	-K*nJ_dc	m/s	Mesh velocity, n direction
☐ vt	0	m/s	Mesh velocity, t direction

MESH GENERATION

1 Click the **Initialize Mesh** button.

2 Click the **Refine Mesh** button once.

COMPUTING THE SOLUTION

1 From the **Solve** menu, select **Solver Parameters**.

2 In the **Time stepping** area, type `0:1:10` for **Times**. This defines a simulation that runs from 0 to 10 s in steps of 1 s.

3 Click **OK** to close the **Solver Parameters** dialog box.

4 Click the **Solve** button (equal sign) on the Main toolbar to solve the model (or choose **Solve Problem** from the **Solve** menu).

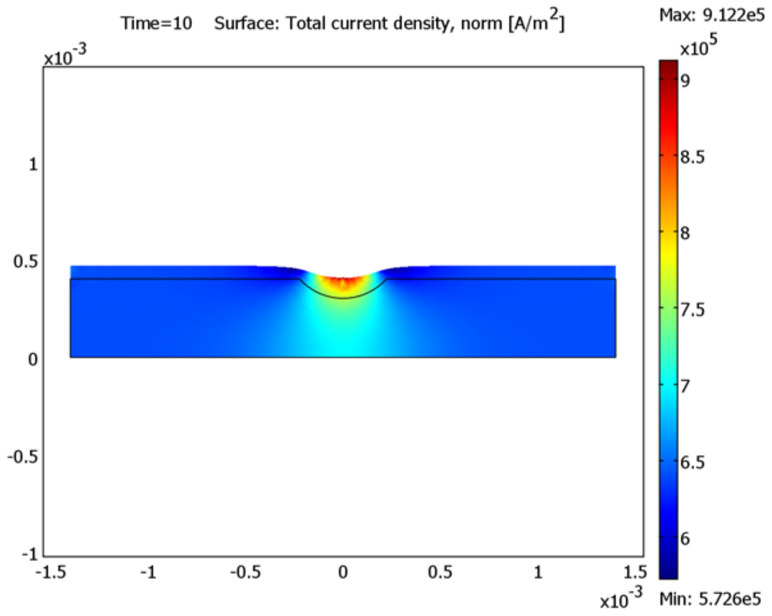
POSTPROCESSING AND VISUALIZATION

The default plot shows the potential distribution on the deformed mesh. Change the visualization settings to plot the current density distribution:

1 From the **Postprocessing** menu, select **Plot Parameters**.

2 On the **Surface** page, select **Conductive Media DC (dc)>Total current density, norm** from the **Predefined quantities** list.

3 Click **OK** to plot.

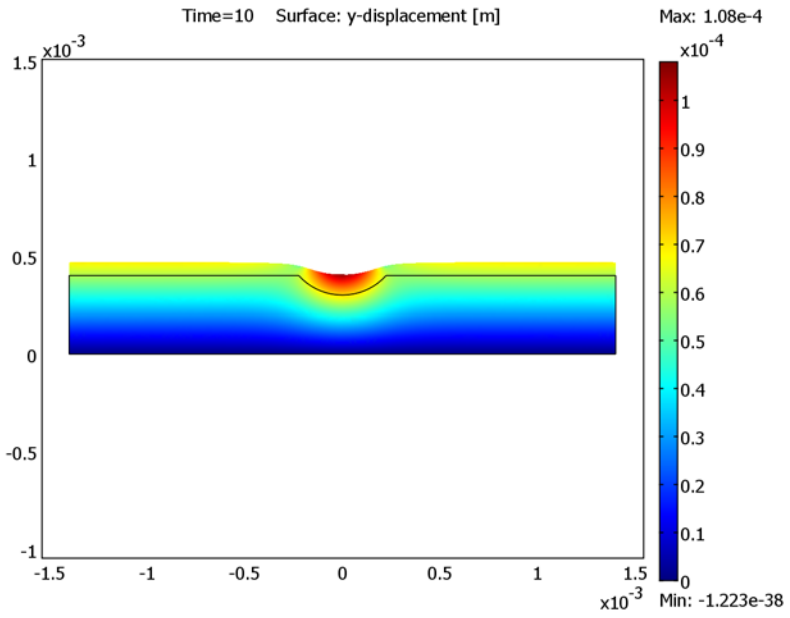


The maximum current density appears to be of the order of 10^6 A/m^2. To see the magnitude of the depletion in the *y* direction more easily, plot the ALE-displacement variable *dy*:

1 From the **Postprocessing** menu, select **Plot Parameters**.

2 On the **Surface** page, type `dy_ale` in the **Expression** edit field.

3 Click **OK** to plot.



The maximum value for the y-displacement is approximately 10^{-4} m (or 0.1 mm).

You can now compare this with an approximate formula for the total depletion increment, d:

$$d = |U|\Delta t = K|J_n|\Delta t = \left(10^{-11}\,\frac{\mathrm{m}^3}{\mathrm{As}}\right)\cdot\left(10^{6}\,\frac{\mathrm{A}}{\mathrm{m}}\right)\cdot(10^{1}\mathrm{s}) = 10^{-4}\ \mathrm{m}$$

This shows that the approximate formula (which does not take effects from the curved boundary into account) is in fact very accurate.

The Parameterized Geometry Application Mode

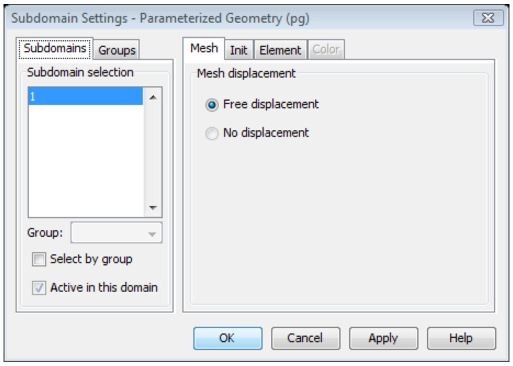
The Parameterized Geometry application mode lets you study how the physics changes when the geometry changes as a function of a parameter. The geometry cannot change dynamically. Rather you have a series of fixed geometries of different shapes.

The Parameterized Geometry application mode uses the same technique to calculate the mesh displacement as the Moving Mesh (ALE) application mode. The difference between the two application modes is the available settings on subdomains and boundaries.

See also the "The Moving Mesh Application Mode" on page 401 for more information about deformed meshes. The sections about frames, smoothing methods, and remeshing also applies to the Parameterized Geometry application mode.

Subdomain Settings

The **Subdomain Settings** dialog box in this application mode contains properties for the mesh displacement.


Subdomain Settings - Parameterized Geometry (pg)

Subdomain selection: 1

Group:

Select by group [ ]

Active in this domain [✓]

Mesh displacement

Free displacement [✓]

No displacement [ ]

Figure 14-9: The Subdomain Settings dialog box of the Parameterized Geometry application mode.

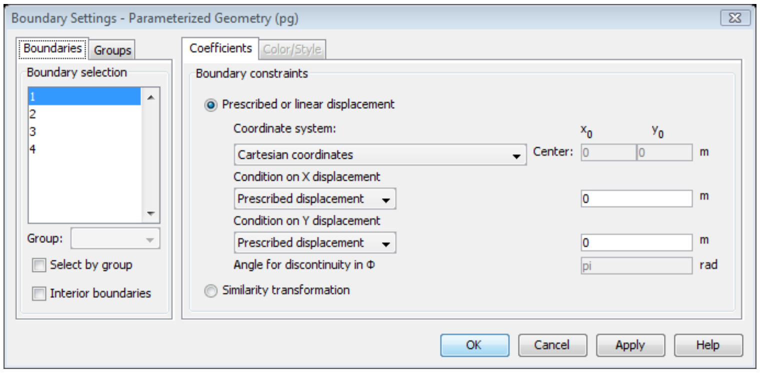
On the **Mesh** page you can specify how the displacement of the mesh is computed in each subdomain. The available options are:

- **Free displacement**. The default value. Use this option in domains that change their shapes as a function of the geometrical parameter. The mesh displacement is constrained only by the boundary conditions on the surrounding boundaries. The software obtains the displacement in the subdomain by solving Winslow's smoothing PDE (see "Smoothing Methods" on page 404).
- **No displacement**. Use this in domains that are not affected by the geometrical parameter.

The **Init** and **Element** pages in the same as in the Moving Mesh (ALE) application mode (see "Subdomain Settings" on page 401).

Boundary Conditions

To set the boundary conditions, use the **Boundary Settings** dialog box.



Boundary selection: 1 (selected), 2, 3, 4

Group:

Select by group

Interior boundaries

Boundary constraints

(●) Prescribed or linear displacement

Coordinate system: Cartesian coordinates

Center: x_0 0 y_0 0 m

Condition on X displacement: Prescribed displacement 0 m

Condition on Y displacement: Prescribed displacement 0 m

Angle for discontinuity in Φ: pi rad

( ) Similarity transformation

Figure 14-10: The Boundary Settings dialog box of the Parameterized Geometry application mode.

The boundary conditions in this application mode fall into three categories:

- **Prescribed displacement**. Choose this option to define the displacement as an expression.
- **Linear displacement**. Choose this option when you have prescribed the displacement of the boundary's end points. The linear displacement then ensures that you get a smooth deformation of the mesh between the end points. The displacement can be a linear function of X, Y, the curve parameter s, the angular coordinate φ, or the radial coordinate r (the latter two are only available when you select **Polar**

coordinates). For example, if you choose the displacement $dx = x - X$ to be **Linear in X**, then the software imposes the constraint

$$dx = dx_1 + (dx_2 - dx_1)\frac{X - X_1}{X_2 - X_1}$$

along the boundary, where $dx = dx_1$ at one end point $X = X_1$ and $dx = dx_2$ at the other end point $X = X_2$.

- **Free**. Choose this option if you want the boundary to move freely without any constraint on the displacement.
- **Similarity transformation**. Choose this option when you have prescribed the displacements of the boundary's end points. The software translates, rotates, and scales the boundary so that the constraints at the end points are satisfied. If dx and dy are the displacements in the x and y directions, dx_1 and dy_1 are the values at $X = X_1$, and dx_2 and dy_2 the values at $X = X_2$, then the similarity transformation is

$$\begin{aligned} dx &= a(X - X_1) - b(Y - Y_1) + dx_1 \\ dy &= b(X - X_1) + a(Y - Y_1) + dy_1 \\ a &= \frac{(dx_2 - dx_1)(X_2 - X_1) + (dy_2 - dy_1)(Y_2 - Y_1)}{(X_2 - X_1)^2 + (Y_2 - Y_1)^2} \\ b &= \frac{-(dx_2 - dx_1)(Y_2 - Y_1) + (dy_2 - dy_1)(X_2 - X_1)}{(X_2 - X_1)^2 + (Y_2 - Y_1)^2} \end{aligned}$$

For the prescribed and linear-displacement boundary conditions, you can choose condition on the two displacement components independently.

In the **Coordinate system** list you choose the coordinate system for the mesh displacements in the prescribed and linear-displacement boundary conditions:

- **Cartesian coordinates**. Choose this option to specify conditions on the Cartesian mesh displacements $dx = x - X$ and $dy = y - Y$.
- **Polar coordinates**. Choose this option to specify the center point (x_0, y_0) of a polar coordinate system with radial coordinate R and angular coordinate Φ. You can also specify an angle from the x-axis where the angular coordinate will be discontinuous. This makes it possible to specify conditions on the radial displacement $dr = r - R$ and the angular displacement $d\varphi = \varphi - \Phi$, where r and φ are the values of the polar coordinates in the deformed configuration.
- **Tangent and normal coordinate system in reference mesh**. Choose this option to specify conditions on the displacements tangential and normal to the reference

boundary. That is, you can specify conditions on the tangential displacement `tX*dx+tY*dy` and the normal displacement `nX*dx+nY*dy`.

- **Tangent and normal coordinate system in deformed mesh**. Choose this option to specify conditions on the displacements tangential and normal to the deformed boundary. That is, you can specify conditions on the tangential displacement `tx*dx+ty*dy` and the normal displacement `nx*dx+ny*dy`.

Point Settings

In the **Point Settings** dialog box it is possible to constrain the displacement of points in the geometry. You can choose to specify the displacements in Cartesian or polar coordinates.

Example: Eigenmode Analysis of an L-Shaped Membrane with Rounded Corner

Introduction

Cleve Moler studied the eigenvalues of the L-shaped membrane in his Ph.D. thesis (Ref. 1). Extend the problem by introducing a rounded corner in the concave corner and plot the radius versus the eigenvalue of the rounded corner. COMSOL Multiphysics makes it possible to easily parameterize the problem in terms of the radius of the rounded corner.

Model Definition

This model is a study of how the eigenvalues of the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

change as a function of the radius of the rounded corner. To use the nonlinear parametric solver you need to add a normalizing condition that determines the size of the solution. In this model, it is the integral condition

$$\int_{\Omega} |u|^2 dx\, dy = 1$$

Modeling in COMSOL Multiphysics

Use an integration coupling variable to compute the norm and an ordinary differential equation (ODE) to enter the equation for the constraint on the norm. The value of the Lagrange multiplier λ corresponds to the eigenvalue of the problem. You can also use this technique to solve nonlinear eigenvalue problems such as $\Delta u = \lambda \sin(u)$.

Parameterize the radius of the rounded corner by a parameter p. Initially draw the geometry with a rounded corner with radius 0.5. Scale the straight boundaries adjacent to the rounded corner in the x and y directions, respectively. Scale the rounded corner by a similarity transformation. Drive the parameterization by introducing point constraints involving the parameter p in the vertices adjacent to the rounded corner.

Use the COMSOL Multiphysics eigenvalue solver to get the initial values for the parametric study. Use the parametric solver to track the eigenmodes and eigenvectors from a parameter p from 0 to 0.5. Then, finally, match the original eigenmodes of the L-shaped membrane with the ones derived by the parametric study.

Results

It is possible to continuously deform the first eigenmodes of the problem with a rounded corner to eigenmodes of the problem with a sharp corner. The 4th mode of the problem with rounded corner can be deformed to the 5th mode of the problem with sharp corner, and vice versa. The 8th and 9th modes of the problem with rounded corner merge into a multiple mode for the problem with the sharp corner.

TABLE 14-1: CORRESPONDENCE BETWEEN EIGENVALUES AND EIGENMODES OF THE PROBLEM WITH ROUNDED AND SHARP CORNERS

EIGENVALUE NUMBER	ROUNDED CORNER, RADIUS = 0.5	SHARP CORNER	MAPPING: ROUNDED TO SHARP
1	8.42	9.80	$1 \rightarrow 1$
2	14.9	15.2	$2 \rightarrow 2$
3	19.6	19.8	$3 \rightarrow 3$
4	28.7	29.7	$4 \rightarrow 5$
5	29.6	32.2	$5 \rightarrow 4$
6	39.4	41.9	$6 \rightarrow 6$
7	43.5	45.4	$7 \rightarrow 7$
8	48.8	49.6	$8 \rightarrow (8,9)$
9	49.8	49.6	$9 \rightarrow (8,9)$

Reference

1. C.B. Moler, "Finite Difference Methods for the Eigenvalues of Laplace's Operator," STAN-CS-65-22, Stanford University, 1965.

Model Library path:

```
COMSOL_Multiphysics/Benchmarks/lshaped_membrane_parametric
```

Modeling Using the Graphical User Interface

MODEL NAVIGATOR

1 Select **2D** from the **Space dimension** list.

2 In the application mode list, open **COMSOL Multiphysics>Deformed Mesh** and then **Parameterized Geometry**.

3 Click the **Multiphysics** button and then the **Add** button.

4 In the application mode list, open **COMSOL Multiphysics>PDE Modes** and then double-click **PDE, Coefficient Form**. Make sure that the **PDE, Coefficient Form** application mode appears together with the **Parameterized Geometry** application mode under **Frame (pg)**.

5 Click **OK**.

OPTIONS AND SETTINGS

1 From the **Options** menu, choose **Axes/Grid Settings**.

2 Click the **Grid** tab, clear the **Auto** check box, and add **Extra x:** `-0.5` and **Extra y:** `0.5`.

3 Click **OK**.

GEOMETRY MODELING

1 On the **Draw** toolbar, click **Line**.

2 Click (-1, 0), (-1, -1), (1, -1), (1, 1), (0, 1), (0, 0.5).

3 On the **Draw** toolbar, click **2nd Degree Bézier Curve**.

4 Click (0, 0), (-0.5, 0).

5 Close the geometry object by right clicking the mouse.

6 Click the **Zoom Extents** button on the Main toolbar.

PHYSICS SETTINGS

Subdomain Settings

1 Make sure the **PDE, Coefficient Form (c)** application mode is selected on the **Multiphysics** menu.

2 On the **Physics** menu, open the **Subdomain Settings** dialog box.

3 Select Subdomain 1.

4 Enter the source term `lm*u` in the **f** coefficient edit field.

5 On the **Init** tab, enter the initial value `u` in the **u(t_0)** edit field.

Coupling Variables

1 From the **Options** menu, choose **Integration Coupling Variables>Subdomain Variables** to open the **Subdomain Integration Variables** dialog box.

2 On Subdomain 1, define the variable `uint` as the expression `u^2` using integration order 4. Make sure the **Global destination** check box is selected.

3 Click **OK**.

Global Equations

1 On the **Physics** menu, open the **Global Equations** dialog box.

2 On the first row in the table on the **States** tab, type `lm` in the **Name (u)** column and `1-uint` in the **Equation** column. Set its initial value to `lambda` in the **Init (u)** column.

3 Click **OK**.

Boundary Conditions

1 In the **Multiphysics** menu, select the **Parameterized Geometry (pg)** application mode.

2 Open the **Boundary Settings** dialog box on the **Physics** menu.

3 Keep the default setting on Boundaries 1, 2, 5, and 6.

4 On Boundary 3, select **Prescribed or linear displacement**, **Linear in X** for **Condition on X displacement**, and **Prescribed displacement**, `0`, for **Condition on Y displacement**.

5 On Boundary 4, select **Prescribed or linear displacement**, **Prescribed displacement**, `0`, for **Condition on X displacement**, and **Linear in Y** for **Condition on Y displacement**.

6 Select **Similarity transformation** for the rounded corner, Boundary 7.

7 Click **OK**.

Point Settings

1 Open the **Point Settings** dialog box on the **Physics** menu.

2 Select Point 3, and check **X-displacement**, and set `p` as its value.

3 Select Point 4, and check **Y-displacement**, and set `-p` as its value.

4 Click **OK**.

POSTPROCESSING AND VISUALIZATION

1 Open the **Plot Parameters** dialog box.

2 Click the **Surface** tab. Select **PDE, Coefficient Form (c)>u** from the **Predefined quantities** list on the **Surface Data** page.

3 Click **OK**.

COMPUTING THE SOLUTION

1 Open the **Solver Parameters** dialog box.

2 Select the **Eigenvalue** solver.

3 Click **OK**.

4 Open the **Solver Manager** dialog box.

5 On the **Solve For** page, select only the degree of freedom variable u.

6 Click **OK**.

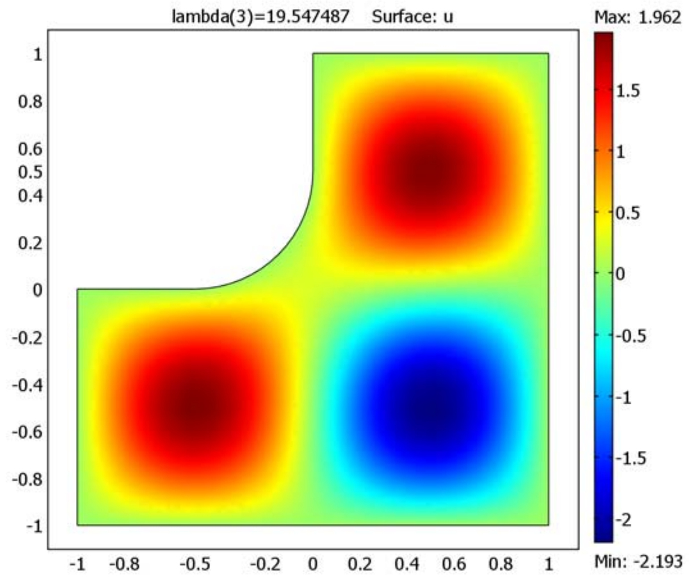
7 Click the **Solve** toolbar button.

POSTPROCESSING AND VISUALIZATION

1 Open the **Plot Parameters** dialog box.

2 On the **General** page, select the third eigenvalue (approximately 19.5) from the list.

3 Click **OK**.



COMPUTING THE SOLUTION

1 Open the **Solver Manager** dialog box.

2 On the **Initial Value** page, make sure that the **Initial value expression evaluated using current solution** button is selected.

3 Select the third eigenvalue from the **Eigenvalue** list.

4 On the **Solve For** page, press Ctrl+A to select all variables: `u`, `lm`, `x`, and `y`.

5 Click **OK**.

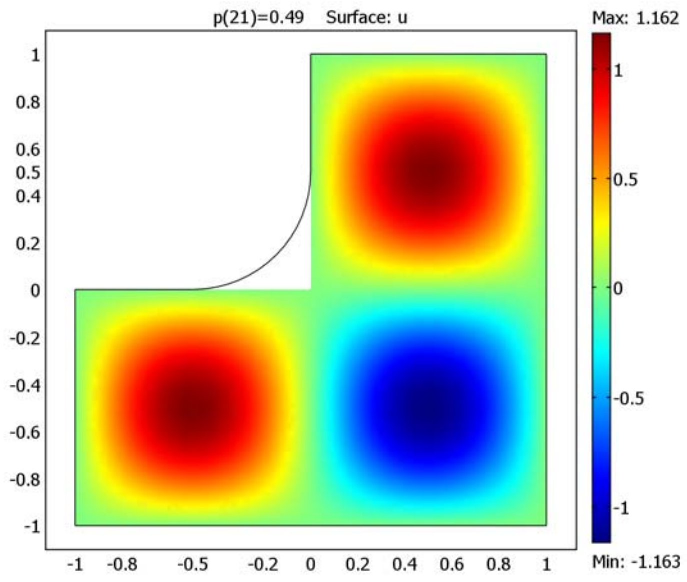
6 Open the **Solver Parameters** dialog box.

7 Select the **Parametric** solver.

8 Set the **Parameter name** to `p`, and the **Parameter values** to `linspace(0,0.49,21)`.

9 Click **OK**.

10 Click the **Solve** button on the Main toolbar.

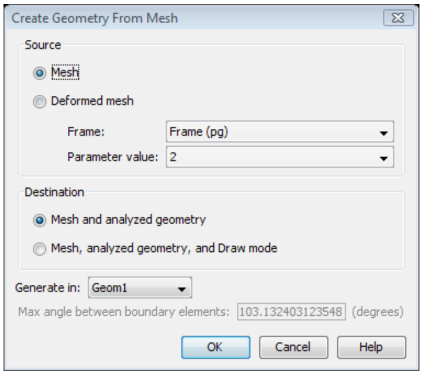


POSTPROCESSING AND VISUALIZATION

Click the **Animate** button on the Plot toolbar to see an animation of the parameterization.

Retrieving the Deformed Mesh

You can retrieve a deformed mesh by selecting **Create Geometry From Mesh** from the **Mesh** menu.


Create Geometry From Mesh

Source

● Mesh

○ Deformed mesh

Frame: Frame (pg)

Parameter value: 2

Destination

● Mesh and analyzed geometry

○ Mesh, analyzed geometry, and Draw mode

Generate in: Geom1

Max angle between boundary elements: 103.132403123548 (degrees)

Figure 14-11: The Create Geometry From Mesh dialog box.

To retrieve a deformed mesh, select **Deformed Mesh** in the **Source** frame. The **Frame** and **Parameter value** or **Solution at time** drop-down lists let you select which source frame and solution to retrieve the deformed mesh from.

In the **Destination** frame you can select if you want to create a mesh and an analyzed geometry or a mesh, an analyzed geometry and a draw mode object from the deformed mesh. Use the **Generate in** drop-down list to select which geometry to put the deformed mesh in.

In the edit field **Max angle between elements in planar face** you specify the maximum allowed angle between two boundary elements to be part of the same planar face. This parameter is only available if the mesh was initially imported and has no parametrization.

Limitations of the Deformed Mesh Feature

The following limitations apply to the deformed mesh feature:

- The connectivity of the mesh remains unchanged during the mesh deformation, which means that topological changes in the geometry are not allowed.
- When the mesh deformation becomes large, the quality of the mesh can deteriorate. If this happens, the solver runs into convergence problems. Sometimes you see the warning **Inverted mesh element** in the Progress window for the solver, which means that a mesh element has (partially) warped inside-out. Use a smaller mesh deformation if you get these problems.
- If you work with finite elements of types other than Lagrange, solving for the mesh deformation coupled to the non-Lagrange elements can result in convergence problems. However, solving first for the mesh deformation and then for the physics in a one-way coupled fashion works. The reason for this is that if `u` is discretized with non-Lagrange elements, first-order spatial derivatives like `ux` does not give a contribution to the Jacobian matrix from the dependence on the mesh motion. Moreover, second-order spatial derivatives never make such Jacobian contributions.
- Solving time-dependent wave equations on a moving mesh does not work, because the formula for the second-order spatial time derivative `utt` in terms of reference time derivatives is not implemented.
- When using second-order (or higher-order) elements in the deformed mesh application modes, the mesh moving techniques often produce elements with distorted shapes. The measure of mesh quality does not capture these distorted shapes because it is computed from the positions of the corners of the mesh element (ignoring mid-side nodes, for instance). For these reasons, it is often best to use linear elements for the mesh positions in the deformed mesh application modes. You can select element type on the **Element** tab in the **Subdomain Settings** dialog box for the Moving Mesh (ALE) and Parameterized Geometry application modes. You must also open the **Model Settings** dialog box from the **Physics** menu and select the ALE frame (typically **Frame (ale)**) in the **Frames** list and select the appropriate element order (typically **Linear**) in the **Geometry shape order** list.

15

Stabilization Techniques

This chapter explains the different stabilization techniques that are available to stabilize the solution to convection-dominated transport problems.

Numerical Stabilization

Modeling transport phenomena for complex geometries usually requires numerical solutions based on some kind of discretization, and in COMSOL Multiphysics' case in the form of finite elements. The discretization of convection-dominated transport problems can introduce instabilities in the solution. You might detect these instabilities as oscillations in the studied fields, primarily where steep gradients are present. The oscillations can even be large enough to prevent the solution from converging. To address these problems, COMSOL Multiphysics includes several stabilization techniques. This section explains these techniques and their modes of operation.

Introduction

Consider a typical scalar transport problem of a convection and diffusion type governed by the following equation and boundary conditions:

$$\begin{array}{ll} \nabla \cdot (-c\nabla u + \beta u) = 0 & \text{in } \Omega \\ -\mathbf{n} \cdot (-c\nabla u + \beta u) = g & \text{on } \partial\Omega_1 \\ u = r & \text{on } \partial\Omega_2 \end{array}$$

This formulation can represent a mass balance or energy balance in a fluid. You can derive some information on the general nature of the solution from dimensionless numbers that describe the relative influence of the different terms in the equation:

- The Peclet number, Pe, is a dimensionless number that describes the relationship between the convective and diffusive terms in a convection and diffusion equation.
- For fluid mechanics problems, where diffusive terms are introduced through the viscosity, the same information is contained in the Reynolds number, Re.

The definitions of both numbers also contain a typical length scale.

You can calculate the Peclet and Reynolds numbers both locally and globally, depending on which length scale is used in their definition. In the same way that the global Peclet and Reynolds numbers influence the solution's global properties and stability, their local counterparts affect the stability of the numerical solution scheme.

The local definition of the Peclet number includes the product of the local mesh size, h, and the magnitude of the convective velocity, β, divided by the diffusivity, c, that is:

$$\mathrm{Pe}_{\mathrm{cell}} = \frac{h|\beta|}{c}$$

The local Reynolds number is the ratio of inertia to viscosity, multiplied by the local mesh size:

$$\mathrm{Re}_{\mathrm{cell}} = \frac{h\rho|v|}{\eta}$$

Where applicable, these dimensionless numbers are available as postprocessing variables, for example, `cellRe_ns`, which is the cell Reynolds number in the Incompressible Navier-Stokes application mode, if you use the default application mode name. As mentioned earlier, the Pe and Re numbers are measures of the relation between directed convection and diffusion. Oscillations can occur where any of the following conditions exist and the Pe or Re number exceeds 2:

- A Dirichlet boundary condition can force the solution to take on a steep gradient near the boundary forming a boundary layer. If the mesh cannot resolve the boundary layer, this creates a local disturbance.
- An initial condition varying in space (one that the mesh does not resolve) causes a local initial disturbance that over time propagates through the computational domain.
- A small initial diffusion term close to a varying source term or varying Dirichlet boundary condition can result in a local disturbance.
- Transport of a varying field or layer that the mesh does not resolve causes a transient under-resolved area.

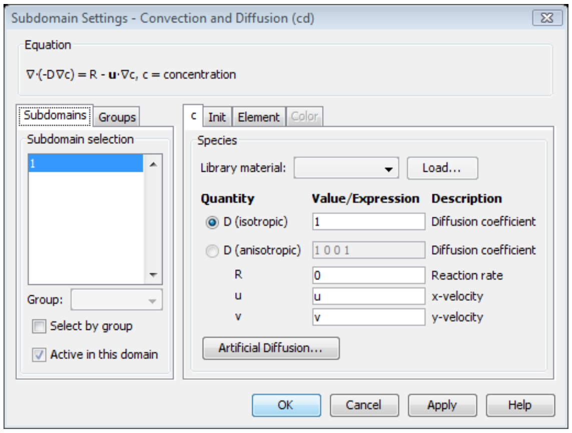
Clearly all of these problems result from the poor discretization of the continuous equations and are not general properties of the equations themselves. As long as any diffusion is present in the problem, there is—at least in theory—a mesh resolution beyond which the discretization is stable. In most cases the mesh size is determined by available memory in the computer used for the simulation, which implies that a given mesh cannot always properly resolve small-scale effects.

Application Modes and Artificial Diffusion

In COMSOL Multiphysics and the Chemical Engineering Module, Earth Science Module, Heat Transfer Module, and MEMS Module, the application modes for fluid dynamics, convective heat transfer, and convective mass transfer provide predefined stabilization through artificial diffusion.

The stabilization effect is established through the addition of weak-form contributions to the original equations.

To access these features in application modes that support artificial diffusion, click the **Artificial Diffusion** button in the **Subdomain Settings** dialog box.


Subdomain Settings - Convection and Diffusion (cd)

Equation

∇·(-D∇c) = R - **u**·∇c, c = concentration

Subdomain selection: 1

Group:

Select by group [ ]

Active in this domain [✓]

Species

Library material:

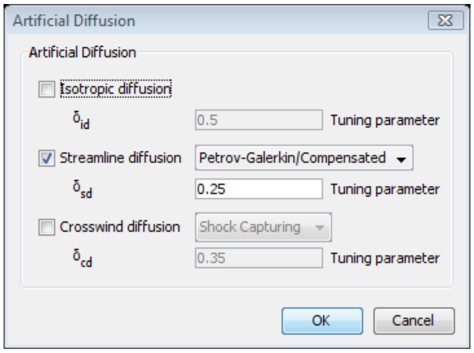
Quantity	Value/Expression	Description
● D (isotropic)	1	Diffusion coefficient
○ D (anisotropic)	1 0 0 1	Diffusion coefficient
R	0	Reaction rate
u	u	x-velocity
v	v	y-velocity

Figure 15-1: The Subdomain Settings dialog box for Convection and Diffusion includes am Artificial Diffusion button.

In the **Artificial Diffusion** dialog box you can select from as many as four artificial diffusion methods:

- Isotropic diffusion/Turbulent isotropic diffusion (see “Isotropic Diffusion” on page 439)
- Streamline diffusion (see “Streamline Diffusion” on page 440).

- Crosswind diffusion (see “Crosswind Diffusion” on page 445)
- Pressure stabilization (see “Pressure Stabilization” on page 446)


Artificial Diffusion

Artificial Diffusion

[ ] Isotropic diffusion
δ_{id} 0.5 Tuning parameter

[✓] Streamline diffusion Petrov-Galerkin/Compensated
δ_{sd} 0.25 Tuning parameter

[ ] Crosswind diffusion Shock Capturing
δ_{cd} 0.35 Tuning parameter

Figure 15-2: The Artificial Diffusion dialog box for the Convection-Diffusion application mode.

Which of the artificial diffusion methods that are available depend on the application mode. The pressure stabilization is only available in the application modes that are based on the Navier-Stokes equations (such as the Incompressible Navier-Stokes, Non-Isothermal Flow, and Non-Newtonian Flow application modes). The Euler Flow and Bubbly Flow application mode include the isotropic diffusion and streamline diffusion methods. See the documentation of the application modes for information about which types of artificial diffusion they support.

Some of the artificial diffusion methods include a tuning parameter, the δ parameter, which controls the amount of artificial diffusion that COMSOL Multiphysics adds. Ideally, tune δ specifically to the problem in question and let it depend on the shape function’s polynomial order. A good starting value for streamline diffusion is 1/(2 times the element order), and therefore δ_{sd} has 0.25 as its default, which is a setting that corresponds to second-order elements (the default for most dependent variables). Turek (Ref. 1) discusses this value and indicates practical values of δ in the range [0.1…2]; however, he apparently makes no conclusion about an optimum and universal value.

COMSOL Multiphysics adds the stabilization terms through the weak-form coefficients instead of directly to the diffusion coefficients. You can check these stabilization terms in the **weak** and **dweak** edit fields. To do so:

1 On the **Physics** menu, point to **Equation System**, and then click **Subdomain Settings**.
2 Click the **Weak** tab in the **Subdomain Settings** dialog box for the equation system.

3 Examine the weak-form contributions from the artificial diffusion in the **weak** and **dweak** edit fields.

Selecting an Artificial Diffusion Method

A general guideline when using artificial diffusion is to first use streamline diffusion using the Galerkin least-squares (GLS) method. The GLS method is active by default in most fluid-flow application modes in COMSOL Multiphysics, the Chemical Engineering Module, and the Heat Transfer Module.

Streamline diffusion can produce very sharp solutions but cannot prevent undershoots and overshoots at sharp gradients. Crosswind diffusion, preferably with shock capturing, addresses exactly this.

If all else fails, try pure isotropic diffusion, which stabilizes most problems of the convection-diffusion type.

Use the pressure stabilization option only when you want to use equal-order interpolation for Stokes type of problems, thus circumventing the Babuska-Brezzi condition (the Inf-Sup stability criterion). Some tuning might be required to get the incompressibility constraint satisfactorily fulfilled.

Adding Artificial Diffusion Manually

Sometimes you might need to use stabilization techniques even though no application mode with this functionality is available. Examples of such situations appear in the *COMSOL Multiphysics Model Library* in “A Transport Problem” on page 111 and “Shock Tube” on page 230.

Artificial Diffusion Types

Several techniques are available to handle local numerical instabilities without the need for mesh refinement. These techniques can all be grouped under the name *artificial diffusion* (artificial viscosity or numerical diffusion/viscosity are also common names for these techniques). The following sections explain the stabilization methods in COMSOL Multiphysics.

Isotropic Diffusion

In this approach, the software adds isotropic diffusion to the equations. It adds a coefficient of artificial diffusion,

$$c_{\mathrm{art}} = \delta_{\mathrm{id}} h|\beta|,$$

where δ_{id} is a tuning parameter (the default value is 0.5).

to the diffusion already present in the problem. You can now express the new local Peclet number as

$$\mathrm{Pe}_{\mathrm{cell}} = \frac{h|\beta|}{(c + c_{\mathrm{art}})} = \frac{2h|\beta|}{2c + h|\beta|}.$$

Clearly $\mathrm{Pe}_{\mathrm{cell}}$ approaches but never exceeds 2 when β approaches infinity. It should be stressed that artificial diffusion is present only in the areas where it is needed because it depends on both h and β.

The downside of this approach is that it does not solve the original problem; instead, you solve the modified $O(h)$ perturbed problem:

$$\begin{aligned} \nabla \cdot (-(c + c_{\mathrm{art}})\nabla u + \beta u) &= 0 \quad \text{in } \Omega \\ -\mathbf{n} \cdot (-(c + c_{\mathrm{art}})\nabla u + \beta u) &= g \quad \text{on } \partial\Omega_1 \\ u &= r \quad \text{on } \partial\Omega_2 \end{aligned}$$

The solution might not be satisfactory in all cases, but the added diffusion definitely damps the effects of oscillations and impedes their propagation to other parts of the system.

Streamline Diffusion

Streamline diffusion adds artificial diffusion only along the streamlines only. COMSOL Multiphysics includes four types of streamline diffusion, which you select from the **Streamline diffusion** list:

- Anisotropic Diffusion—see "Anisotropic" below
- Galerkin Least-Squares (GLS)—see
- Petrov-Galerkin—see "Petrov-Galerkin (SUPG)" on page 441
- Petrov-Galerkin/Compensated—see "Petrov-Galerkin (SUPG)" on page 441

ANISOTROPIC

In many cases, there is no need for any artificial diffusion in the direction orthogonal to the main flow. It is therefore possible to modify c_{art} to a form that only adds artificial diffusion in the flow—or streamline—direction.

The normal interpretation of the diffusivity in the original equation is as a scalar number, but it can also be a diagonal tensor. This implies that you can also define the coefficient of artificial diffusion as a tensor. The natural choice is to define c_{art} as a rank-one tensor

$$c_{\text{art},\,ij} = \frac{\delta h \beta_i \beta_j}{|\beta|}$$

where δ is a tunable parameter that controls the amount of artificial diffusion. If you define c_{art} in this way it has only one nonzero eigenvalue $c_\beta = \delta h|\beta|$, whose corresponding eigenvector lies in the streamline direction. If you select $\delta = 0.5$, the magnitude of the artificial diffusion in the streamline direction (c_β) is the same as in the classic case described earlier.

COMSOL Multiphysics implements the anisotropic artificial diffusion in a slightly different way. Instead of modifying the actual diffusivity, the application mode introduces a weak contribution

$$\int_\Omega \frac{\delta h}{|\beta|}(\beta \cdot \nabla v)(\beta \cdot \nabla u)d\Omega$$

"Petrov-Galerkin (SUPG)" on page 441 describes the notation in this expression. More information about the weak form and weak-form contributions is available in "The Weak Form" on page 291.

Analogous to the case of classic artificial diffusion, a problem stabilized with anisotropic diffusion is not identical to the original problem. In fact, the exact solution to the original equations does not solve the modified problem. However, the errors introduced by the anisotropic artificial diffusion are far less serious than the ones created by classic artificial diffusion.

PETROV-GALERKIN (SUPG)

Because COMSOL Multiphysics bases the discretization of the equations on the finite element method, it provides an elegant way of adding artificial diffusion without actually perturbing the original equations. This method is more commonly known as the *streamline upwind Petrov-Galerkin* method. Its origin stems from the weak form of the equations.

The weak formulation of the convection and diffusion equation is

$$\int_{\Omega} \nabla v c \nabla u d\Omega - \int_{\Omega} \nabla v \cdot \beta u d\Omega = \int_{\partial\Omega_1} v g ds$$

$$u = r \quad \text{on } \partial\Omega_2$$

where v is an arbitrary function called a test function. In the absence of streamline diffusion, the test function belongs to the same function space as the solution, u.

Now replace the test function v with

$$v + \delta'\beta \cdot \nabla v \qquad (15\text{-}1)$$

where the primed delta, δ', is

$$\delta' = \frac{\delta h}{|\beta|}$$

and δ is a tuning parameter. This introduces artificial diffusion through the test function itself. Because you can choose this function arbitrarily, there is no explicit perturbation of the original equations whatsoever. This is all true in theory, but in practice the solutions might be somewhat smeared out along the streamlines. Even so, solutions obtained by this method are usually far better than those obtained by any of the methods already mentioned.

The following equation gives the weak formulation of a problem with streamline diffusion:

$$\int_{\Omega}\delta'(\beta\cdot\nabla v)(-c\Delta u+\beta\cdot\nabla u)d\Omega+\int_{\Omega}\nabla vc\nabla ud\Omega-\int_{\Omega}\nabla v\cdot\beta ud\Omega = \int_{\partial\Omega_1} vgdS$$

$$u = r \text{ on } \partial\Omega_2$$

The stabilization can be partly understood by studying the term

$$\int_{\Omega}\delta'(\beta\cdot\nabla v)(\beta\cdot\nabla u)d\Omega$$

which corresponds to an artificial diffusion tensor of the form

$$c_{sd,ij} = \delta'\beta_i\beta_j = \frac{\delta h}{|\beta|}\beta_i\beta_j$$

which corresponds exactly to the anisotropic artificial diffusion case presented earlier. The important difference is that stabilization is introduced through the test function and hence that the formulation is consistent.

PETROV-GALERKIN (SUPG), COMPENSATED

As already mentioned, the Galerkin formulation becomes unstable for large $\mathrm{Pe}_{\mathrm{cell}}$. Hence there is really no use for any stabilization when $\mathrm{Pe}_{\mathrm{cell}}$ is small. The Petrov-Galerkin, compensated streamline diffusion method achieves this by replacing δ' in Equation 15-1 with δ' (δ' > 0) where

$$\delta' = \frac{\delta h}{|\beta|}-\frac{c}{|\beta|^2}$$

This definition of δ' completely eliminates the streamline-diffusion contributions in regions where the diffusivity is large enough to produce a small $\mathrm{Pe}_{\mathrm{cell}}$.

The drawback is the this formulation can cause discontinuities and that it introduces additional nonlinearities.

Note: Expressions derived by the current streamline diffusion implementation are valid only for constant PDE coefficients. Fortunately, this does not imply that the stabilization options are useless for highly complex nonlinear equations. In such cases, however, the added diffusion is not of strict Petrov-Galerkin type and there is a risk of an excessive perturbation of the problem.

GALERKIN LEAST-SQUARES STABILIZATION (GLS)

During the years, the Petrov-Galerkin method has been developed and also given rise to new and often superior methods. One of them is the Galerkin least-squares (GLS) stabilization technique.

GLS is a general stabilization technique applicable to a wide range of problems (see Ref. 1). Its theoretical background is that the test function should be chosen to minimize the square of the residual of the equations. By working with matrices rather than only scalar equations, GLS can be formulated for systems of transport equations.

General Formulation of GLS Stabilization

It is possible to formulate GLS stabilization for any set of independent variables for any set of equations of convection-diffusion type. In conservative variables, the system has the form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{K}_j}{\partial x_j} = \frac{\partial \mathbf{F}_j}{\partial x_j} + \mathbf{B}\,, \tag{15-2}$$

where $\mathbf{U}$ is the vector of conservative variables, $\mathbf{K}_j$ is the convective flux in the jth direction, $\mathbf{F}_j$ is the diffusive flux in the jth direction, and $\mathbf{B}$ is the source vector. The length of the vectors is the number of equations. Reformulated in primitive variables, $\mathbf{V}$, Equation 15-2 can be written as

$$\mathbf{A}_0\frac{\partial V_k}{\partial t} + \mathbf{A}_j\frac{\partial \mathbf{V}}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\mathbf{D}_{ij}\frac{\partial \mathbf{V}}{\partial x_j}\right) + \mathbf{B}\,, \tag{15-3}$$

where $\mathbf{A}_0 = \partial\mathbf{U}/\partial\mathbf{V}$, $\mathbf{A}_j = \partial\mathbf{K}_j/\partial\mathbf{V}$, and the four-dimensional tensor $\mathbf{D}_{ij}$ is such that

$$\mathbf{D}_{ij}\frac{\partial \mathbf{V}}{\partial x_j} = \mathbf{F}_i\,. \tag{15-4}$$

Written in operator formulation, Equation 15-3 becomes

$$\Upsilon\mathbf{V} = \mathbf{B} \tag{15-5}$$

where

$$\Upsilon = \mathbf{A}_O\frac{\partial}{\partial t} + \mathbf{A}_j\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\mathbf{D}_{ji}\frac{\partial}{\partial x_j}\right) \tag{15-6}$$

Applying the Galerkin least-squares technique, the weak formulation of Equation 15-3 gets the following left-side contribution:

$$\sum \int_Q (\Upsilon^T \mathbf{W}) \tau (\Upsilon \mathbf{V} - \mathbf{B}) dQ \tag{15-7}$$

where Υ^T is defined by

$$\Upsilon^T = \mathbf{A}_0^T \frac{\partial}{\partial t} + \mathbf{A}_j^T \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\mathbf{D}_{ji}^T \frac{\partial}{\partial x_j}\right) \tag{15-8}$$

W is the vector of test functions and the summation is over all elements Q. τ is a matrix accounting for different time scales of the equations.

GLS automatically circumvents the Babushka-Brezzi condition. This can be an important aspect when using GLS with geometric multigrid solvers. Actually, in case of linear elements GLS becomes identical to SUPG (see "Petrov-Galerkin (SUPG)" on page 441) in conjunction with pressure stabilization.

For more details, see Ref. 2.

Primitive Pressure Variables

The conservative variables for Navier-Stokes equations are

$$\mathbf{U} = [\rho, \rho u, \rho v, \rho w, \rho e_0] \tag{15-9}$$

where ρ is the density, u, v, w is the velocity field, and e_0 is the total internal energy. Because Equation 15-2 is ill-posed in the incompressible limit, another set of variables must be chosen such that Equation 15-3 does not become singular. COMSOL Multiphysics solves for the so-called pressure variables

$$\mathbf{V} = [p, u, v, w, T] \tag{15-10}$$

where p is the mechanical pressure, and T is the absolute temperature. The matrix τ is for Mach number much smaller than one given by $\tau = \mathrm{diag}(\tau_\mathrm{c}, \tau_\mathrm{m}, \tau_\mathrm{m}, \tau_\mathrm{m}, \tau_\mathrm{e})$ where

$$\tau_c = \frac{|\mathbf{u}|h}{2}\min(1, \mathrm{Re}^h) \tag{15-11}$$

$$\tau_m = \min\left(\frac{\Delta t}{\rho}, \frac{h}{2\rho|\mathbf{u}|}, \frac{\mathrm{m}h^2}{4\eta}\right) \tag{15-12}$$

$$\tau_e = \min\left(\frac{\Delta t}{\rho \mathrm{c_V}}, \frac{h}{2\rho \mathrm{c_V}|\mathbf{u}|}, \frac{\mathrm{m}h^2}{4\kappa}\right) \tag{15-13}$$

where Re^h is the element Reynolds number

$$\mathrm{Re}^h = \frac{\rho|\mathbf{u}|h}{\eta} \tag{15-14}$$

and $\mathrm{m} = \min(1/3, 2\mathrm{C}^{\mathrm{e}})$ where is C^{e} a constant arising from an inverse estimate of the second derivatives of the shape functions (Ref. 3). In Ref. 4, some values for C^{e} have been derived for Lagrange elements for some simple cases. To specify an appropriate value for C^{e} is however a very intricate matter and is possible only by editing the subdomain equations. For usability, COMSOL uses $\mathrm{m} = 1/3$ which is exact for most applications when linear elements are used.

For stationary calculation, terms involving the time step, Δt, should be omitted from Equation 15-12 and Equation 15-13.

Scalar Transport Equations

In the case of transport of a scalar property, s, the time coefficient is given by

$$\tau_s = \min\left(\frac{\Delta t}{\rho}, \frac{h}{2\rho|\mathbf{u}|}, \frac{\mathrm{m}h^2}{4k}, \left|\frac{s}{B}\right|\right) \tag{15-15}$$

where k is the diffusivity of s, and B is the source term. The last argument should be included only if the source can be considered to be strong. See, for example, Ref. 5.

Crosswind Diffusion

Crosswind diffusion adds artificial diffusion in the crosswind direction, that is, the direction perpendicular to the streamlines. COMSOL Multiphysics offers two types of crosswind diffusion:

- Ordo $h^{3/2}$
- Shock capturing

ORDO $H^{3/2}$

The Ordo $h^{3/2}$ crosswind-diffusion algorithm adds a diffusion coefficient according to Ref. 6:

$$c_{\mathrm{cd}} = \delta_{\mathrm{cd}}(h^e)^{3/2}$$

This method should be used only with linear elements but is very computational inexpensive. The constant δ_{cd} is per default set to 0.35. This value is however only valid for SI units and must be recalculated if other physical units are used.

SHOCK CAPTURING

The basic streamline diffusion method can damp out oscillations in the solution, but it does not prevent overshoots and undershoots close to discontinuities. Various shock-capturing techniques can eliminate these effects. Most such methods add terms to the basic SUPG method, terms that depend on the gradient of the solution. The COMSOL Multiphysics implementation adds shock capturing according to:

$$\sum_{e=1}^{N_{el}} \int_{\Omega^e} \frac{1}{2}\alpha_c^e h^e \frac{|\Re(\phi_h)|}{|\nabla\phi_h|} \nabla\psi_h \cdot \left(I - \frac{1}{|\beta|^2}(\beta \otimes \beta)\right) \cdot \nabla\phi_h d\Omega$$

for a variable ϕ, where $\Re(\phi_h)$ is the equation residual $(\Re(\phi_h) = \nabla\Gamma_\phi - F_\phi)$. The effective diffusion coefficient, $\alpha^e{}_c$, is

$$\alpha_c^e = \max(0, \delta_{\mathrm{cd}} - 1/\gamma_p^e)$$

$$\gamma_p^e = \frac{|\beta_p| h^e}{2D_m}$$

$$\beta_p = \frac{F_\phi}{|\nabla\phi_h|^2} \nabla\phi_h$$

where D_m is the mean diffusion coefficient and F_ϕ is the F term in the equation for ϕ. The tuning coefficient δ_{cd} is set to 0.35, which correlates to second-order basis functions (see Ref. 7 and Ref. 8).

Pressure Stabilization

Pressure stabilization using the Petrov-Galerkin pressure stabilization algorithm (also called PSPG for Pressure Stabilization, Petrov-Galerkin) allows for equal-order interpolation (for example, P1-P1 and P2-P2 basis functions) when solving the incompressible Navier-Stokes equations in its primitive variable formulation (see Ref. 9 and Ref. 10). This stabilization type is implemented as follows:

$$\sum_{e=1}^{N_{el}} \int_{\Omega^e} \delta_{\mathrm{ps}} \tau_{\mathrm{ps}} \frac{1}{\rho} \Re(\mathbf{u}) \cdot \nabla\Psi_h d\Omega$$

where δ_{ps} is a tuning constant (default value: 1). Ψ_h is the test function space for the pressure. τ_{ps} is calculated as follows:

$$\tau_{\mathrm{ps}} = \left[\left(\frac{|\mathbf{u}|}{h^e}\right)^2 + \left(\frac{3\eta}{\rho(h^e)^2}\right)^2\right]^{-\frac{1}{2}}$$

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